PAPER CODE NO.
MATH724

THE UNIVERSITY
of LIVERPOOL

MAY 2007 EXAMINATIONS

Bachelor of Arts	:	Year 3
Bachelor of Science	:	Year 3
Master of Mathematics	:	Year 3
No qualification aimed for	:	Year 1

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments.

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1. (i) Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{2y^2}{(x^2+4)}$$

putting your answer in the form $y = f(x)$.

[6 marks]

(ii) Solve the initial value problem

$$\frac{dy}{dx} - y\tan x = 2\cos x - \sec x$$

for $y(x)$ where $y(0) = 2$.

[6 marks]

(iii) Solve the initial value problem

$$\frac{dy}{dx} = \frac{[\sin x \sin(xy^2) - y^2\cos(xy^2)\cos x]}{2xy\cos(xy^2)\cos x}$$

where $y(\pi/4) = 1$.

[8 marks]

2. Find the solution of the ordinary linear differential equation

$$y'' - 2y' + (1-a^2)y = e^x$$

where the parameter a is real and $a \neq 0$ with the the initial condition $y(0) = 0$ and $y'(0) = 0$.
For the case $a = 0$, determine $y(x)$ subject to the same boundary conditions.
[20 marks]

3. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 2x + y + \cos t$$
$$\frac{dy}{dt} = x + 2y + \sin t$$

when $x(0) = 0$ and $y(0) = 0$.

[20 marks]

4. Solve the ordinary differential equation for $y(x)$

$$x^2y'' \; - \; 9xy' \; + \; 25y \; = \; x^2 \; + \; 2x^2 \ln x$$

with the initial conditions $y(1) = 1$ and $y'(1) = 2$.

[20 marks]

5. The function $u(x, y)$ satisfies the first order partial differential equation

$$x\frac{\partial u}{\partial x} \; + \; (x + y)\frac{\partial u}{\partial y} \; = \; u \; + \; 2x \; + \; y$$

in the region $x > 0$ subject to the boundary conditions $u(x, 0) = x$.
Determine the function $u(x, y)$.

[20 marks]

6. For an even periodic function, with period T, show that the Fourier coefficients b_n are zero for all n.

[4 marks]

The periodic function $f(t)$ is defined by

$$f(t) \; = \; 1 \; + \; \left|\cos\left(\frac{\pi t}{T}\right)\right| \qquad - \; T \; < \; t \; \leq \; T$$

and $f(t + 2T) = f(t)$. Sketch $f(t)$ in $- \; 2T \leq t \leq 2T$ and find its period.
Determine the Fourier series of $f(t)$.

[16 marks]

7. By seeking a solution of the form $u(x, t) \; = \; F(x)G(t)$, find the general solution of the heat equation

$$\frac{\partial u}{\partial t} \; = \; k\frac{\partial^2 u}{\partial x^2}$$

in a horizontal bar of length L where k is a positive constant and the left and right ends are held at temperatures T_0 and T_1.
If the initial temperature distribution is

$$u(x, 0) \; = \; \begin{cases} 0^{\mathrm{O}}\mathrm{C} & 0 \; < \; x \; < \; \frac{1}{2}L \\ 100^{\mathrm{O}}\mathrm{C} & \frac{1}{2}L \; < \; x \; < \; L \end{cases}$$

and the ends are held at 20^{O}C, find the temperature distribution, $u(x, t)$, at time t.

[20 marks]