PAPER CODE NO.
MATH724

THE UNIVERSITY
of LIVERPOOL

SUMMER 2005 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments.

1. (i) Find the general solution of the differential equation

$$\frac{dy}{dx} = y^4(x + \cos x)$$

putting your answer in the form $y = f(x)$.

[5 marks]

(ii) Solve the initial value problem

$$xy^2 \frac{dy}{dx} = x^3 + y^3; \qquad y(1) = 2.$$

[6 marks]

(iii) By forming an exact differential, or otherwise, solve the initial value problem

$$x \cos y \frac{dy}{dx} + 2x + \sin y = 0$$

with $y(1) = 0$.

[9 marks]

2. Solve the differential equation for $z(x)$

$$x^2 z'' - 5xz' + 9z = 25\sqrt{x} + 3\ln x$$

with the initial conditions $z(1) = 2, \quad z'(1) = 0.$

[20 marks]

3. (i) Write down the Cauchy-Riemann equations connecting a function $u(x, y)$ to its conjugate harmonic function $v(x, y)$.

Show that the function

$$u(x, y) = \frac{6xy}{(x^2 + y^2)^2} + 4xy$$

satisfies the two-dimensional Laplace's equation if $x^2 + y^2 \neq 0$.

[**8 marks**]

(ii) Find $v(x, y)$, the conjugate harmonic function corresponding to $u(x, y)$ in part (i).

[**12 marks**]

4. Solve the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 14x - 10y + e^t \\ \frac{dy}{dt} &= 5x - y + e^t \end{aligned}$$

with the initial conditions $x(0) = 1, y(0) = 2$.

[**20 marks**]

5. The function $u(x,y)$ satisfies the first order partial differential equation

$$\frac{\partial u}{\partial x} - e^x \frac{\partial u}{\partial y} = \frac{u}{x}$$

in the domain $y > 0$. On the boundary, $y = 0$, the value of u is given by

$$u(x,0) = 2 \ .$$

(i) Show that the family of characteristics of this partial differential equation may be represented by

$$x = s + t \ , \qquad y = e^s - e^{s+t}$$

where s and t are parameters whose significance you should explain.

[8 marks]

(ii) Hence, or otherwise, determine the function $u(x,y)$.

[12 marks]

6. The function $g(t)$ has period 2π, and it has the value

$$g(t) = t \sin t \qquad \text{for} \quad -\pi < t < \pi.$$

(i) Sketch g in the range $-2\pi < t < 2\pi$. Is this function odd, even or neither?

[4 marks]

(ii) Calculate the Fourier series for $g(t)$.

[16 marks]

Hint: Remember that $\cos A \sin B = \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)$.

7. The temperature $U(\theta, t)$ in a metal ring obeys the heat equation

$$\frac{\partial U}{\partial t} = \kappa \, \frac{\partial^2 U}{\partial \theta^2} \, ,$$

where the angular coordinate θ will be chosen to run from $-\pi$ to π.

(i) By considering the separable solutions of the heat equation show that the general solution to the heat equation in the ring can be written as

$$U(\theta, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\kappa n^2 t} \left[a_n \cos n\theta + b_n \sin n\theta \right]$$

where n is an integer and a_n and b_n are constants.

[**8 marks**]

(ii) Initially the temperature is $100\,^{\circ}\mathrm{C}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $20\,^{\circ}\mathrm{C}$ in the rest of the ring. Find the temperature distribution at later times.

[**12 marks**]