PAPER CODE NO.
MATH724

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted. Marks for parts of questions may be subject to small adjustments.

1. (i) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^y(x^2 + 2),$$

putting your answer in the form $y = f(x)$.

[5 marks]

(ii) Solve the initial value problem

$$\frac{dy}{dx} = 4\frac{y}{x} - x^2 \; ; \quad y(1) = 0 \; .$$

[5 marks]

(iii) By forming an exact differential, or otherwise, solve the initial value problem

$$2\frac{dy}{dx} = \frac{1 - 3x^2y^2}{x^3y}$$

with $y(1) = 2$.

[10 marks]

2. A pendulum driven by the force $f(t)$ obeys the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = f(t).$$

If the force $f(t)$ is a periodic function, with period 2π, satisfying

$$f(t) = t \qquad \text{for} \qquad -\pi < t < \pi$$

show that f can be represented by the series

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin nt$$

Use this result to calculate the Fourier series for the steady-state solution for $y(t)$. You may assume that resonance does not occur.

[20 marks]

3. Solve the differential equation

$$2x^2z'' - xz' + z = x + 1$$

with the initial conditions $z(1) = 0, \quad z'(1) = 1.$

[20 marks]

4. Find the general solution to the system of equations

$$\begin{aligned} \frac{d^2x}{dt^2} &= -4x + y + 2 \\ \frac{d^2y}{dt^2} &= 4x - 4y \end{aligned}$$

[20 marks]

5. The function $u(x, y)$ satisfies the first order partial differential equation

$$(x+1)\frac{\partial u}{\partial x} + 3(y+2)\frac{\partial u}{\partial y} = u + 1$$

in the domain $x > 0$ and the boundary condition

$$u(0, y) = y^2 \quad \text{on} \quad x = 0\ .$$

(i) Show that the characteristic curves for this partial differential equation can be written as

$$x = e^t - 1, \qquad y = (s+2)e^{3t} - 2$$

[**8 marks**]

(ii) Hence, or otherwise, determine the function $u(x, y)$.

[**12 marks**]

6. The function $f(x)$ has period π, and it has the value

$$f(x) = \exp(-\alpha x) \qquad \text{for} \quad 0 < x < \pi.$$

(i) Sketch f in the range $-2\pi < x < 2\pi$.

[**4 marks**]

(ii) Calculate the Fourier series for $f(x)$.

[**16 marks**]

7. A function $u(x, y)$ satisfies Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$ together with the homogeneous boundary conditions

$$u(0, y) = u(a, y) = 0, \quad 0 < y < b$$

on $x = 0$ and $x = a$.

(i) By seeking solutions of the form $F(x)G(y)$ show that the above problem has solutions of the form

$$u(x, y) = \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right]$$

where n is an integer and C_n and D_n are constants.

Hence write down the general solution of Laplace's equation in the rectangle for the given boundary conditions.

[**10 marks**]

(ii) Find the solution to this problem, i.e. find all C_n and D_n, given that $u(x, y)$ satisfies the boundary conditions

$$u(x, 0) = 0, \; u(x, b) = 1, \quad 0 < x < a$$

on $y = 0$ and $y = b$.

[**10 marks**]