MATH724 Summer Exam 2001

Solutions

1. (i) The auxiliary equation is

$$m^2 + 5m + 6 = 0 \qquad \Rightarrow m = -3, -2$$

Hence

$$y_h = C_1 e^{-3t} + C_2 e^{-2t}$$

For y_p we try

$$\begin{aligned} y_p &= A\cos(t) + B\sin(t) \\ \dot{y}_p &= -A\sin(t) + B\cos(t) \\ \ddot{y}_p &= -A\cos(t) - B\sin(t) \end{aligned}$$

Substituting in, gives

$$5(A\cos(t) + B\sin(t)) + 5(-Asin(t) + B\cos(t)) = 20\cos(t)$$

Equating coefficients: $5A + 5B = 20; 5B - 5A = 0$. Hence $A = B = 2$. The general solution is

$$C_1 e^{-3t} + C_2 e^{-2t} + 2(A\cos(t) + B\sin(t))$$

Putting in the initial conditions yields $C_1 + C_2 + 2 = 2; -3C_1 - 2C_2 + 2 = 6$. Hence $C_1 = -C_2,\ C_2 = 4$. Hence the solution is

$$-4e^{-3t} + 4e^{-2t} + 2\cos(t) + 2\sin(t)$$

(ii) Write $z = \mathcal{L}(y)$. Then

$$sz - y(0) \equiv \mathcal{L}(y') = sz - 2.$$

similarly

$$s^2 z - 2s - 6 = \mathcal{L}(y'')$$

Thus

$$(s^2 + 5s + 6)z = -2s - 16 = \mathcal{L}(20\cos(t)) = \frac{20s}{s^2+1}$$

Hence

$$z = \frac{2s+16}{s^2+5s+6} + \frac{20s}{(s^2+1)(s^2+5s+6)}$$

Using partial fractions:

$$\begin{aligned}\frac{2s+16}{s^2+5s+6} &= \frac{12}{s+2} - \frac{10}{s+3}\\ \frac{20s}{(s^2+1)(s^2+5s+6)} &= \frac{6}{s+3} - \frac{8}{s+2} + \frac{D_1 s + D_2}{s^2+1}\end{aligned}$$

where D_1 and D_2 are unknown coefficients (the other coefficients have been obtained using the cover-up rule). Equating coefficients, we get $D_1 = D_2 = 2$. Hence

$$z = \frac{4}{s+2} - \frac{4}{s+3} + 2\frac{s+1}{s^2+1}.$$

Now $\mathcal{L}^{-1}(1/(s+3)) = e^{-3t}$, $\mathcal{L}^{-1}(2/(s^2+1)) = \cos(t)$, $\mathcal{L}^{-1}(1/(s^2+1)) = \sin(t)$. Hence

$$y(t) = 4e^{-3t} + 4e^{-2t} + 2\cos(t) + 2\sin(t)$$

which is the same result.

2. (i) Write $u(x,y) = F(x)G(y)$. Then

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2}$$

As these are independent of x and y, they are equal to a constant. Because of the b.c at $x = 0$ this is $-m^2$. Then

$$\frac{1}{F}\frac{d^2F}{dx^2} = -m^2$$

Now $u(a,y) = 0 \;\Rightarrow \sin(ma) = 0 \Rightarrow ma = n\pi$ where n is an integer. Thus

$$F = \sin\frac{n\pi x}{a}\frac{-1}{G}\frac{d^2G}{dy^2} = -n^2\pi^2/a^2$$

$$\Rightarrow\ G = C_n\cosh\frac{n\pi y}{a} + D_n\sinh\frac{n\pi y}{a}$$

Hence the result.

(ii) The solution is sum over n. We need to find the coefficients C_n and D_n. Now $u(x,0) = 1$: hence

$$\sum_n \sin\frac{n\pi x}{a}\left(C_n\cosh(0) + D_n\sinh(0)\right) = 1$$

i.e.

$$\sum_n C_n\sin\frac{n\pi x}{a} = 1$$

Using Fourier series

$$C_n = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}dx = 0$$

if n is even, or $4/n$ if n is odd. At $y = b$, $u(x,0) = 0$ Hence

$$\sum_n \sin\frac{n\pi}{a}\left(C_n\cosh\frac{n\pi b}{a} + D_n\sinh\frac{n\pi b}{a}\right) = 0$$

Hence

$$D_n = -C_n\coth\frac{n\pi b}{a}$$

Giving finally,

$$u(x,y) = \sum_k \sin\frac{(2k+1)\pi x}{a}\left[\frac{4}{2k+1}\cosh\frac{(2k+1)\pi y}{a} - \coth\frac{(2k+1)\pi b}{a}\sinh\frac{(2k+1)\pi y}{a}\right]$$

3. (i)

$$\begin{aligned}\mathcal{L}(e^{at}) &= \int_0^\infty e^{-st}e^{at}\cdot dt\\ &= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a} \qquad \text{for } s > a\end{aligned}$$

(ii) Here we need to perform an integration by parts of $f'(t)$ yields

$$\begin{aligned}
& \int_0^\infty f'(t)e^{-st}dt \\
= & \left[fe^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\
= & -f(0) + s\mathcal{L}f(t)
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}(f''(t)) &= -f'(0) + s\mathcal{L}(f'(t)) \\
&= -f'(0) - sf(0) + s^2\mathcal{L}(f) \\
&= -f(0) + s\mathcal{L}f(t)
\end{aligned}$$

(iv)

$$\begin{aligned}
\int_s^\infty F(s')ds' &= \int_s^\infty ds' \int_0^\infty e^{-s't}f(t)dt \\
&= \int_0^\infty f(t)dt \int_s^\infty e^{-s't}ds' \\
&= \int_0^\infty f(t)dt \left(-\frac{1}{t}\right)\left[e^{-s't}\right]_s^\infty \\
&= \int_0^\infty \frac{e^{-st}}{t} f(t)dt \equiv \mathcal{L}\left(\frac{f}{t}\right)
\end{aligned}$$

(v) Substituting the results of (ii) and (iii) above into the differential equation given, writing $Y(s) \equiv \mathcal{L}\{y()\}$, and inserting the initial conditions, one obtains

$$s^2Y(s) - s + 2 + 4(sY(s) - 1) + 3Y(s) = (s^2 + 4s + 3)Y(s) = s + 2 + \frac{1}{s+2}$$

Using partial fractions

$$\begin{aligned}
Y(s) &= \frac{1}{s+3} + \frac{1}{s+1} - \frac{1}{s+2} \\
\Rightarrow y(t) &= \exp(-st) + \exp(-t) - \exp(-2t)
\end{aligned}$$

4. (i)

$$f(x) = f(-x)$$

Hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Putting $x = -x$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x')\sin(-nx')dx' \\ &= \frac{-1}{\pi}\int_{-\pi}^{\pi} f(x')\sin(nx')dx' \\ &= -b_n \end{aligned}$$

which can only be true if b_n is zero $\forall n$.

(ii) Firstly, we note that the function is even so $b_n = 0$

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T}^{T} |\sin\left(\frac{\pi t}{T}\right)|dt \\ &= \frac{4}{\pi} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} |\sin(\pi x/T)|\cos(n2\pi x/T)dx \\ &= \frac{4}{T}\int_{0}^{T/2} \sin(\pi x/T)\cos(n2\pi x/T)dx \\ &= \frac{2}{T}\int_{0}^{T/2} \{\sin((2n+1)\pi x/T) - \sin((2n-1)\pi x/T)\}\,dx \\ &= \frac{2}{\pi}\left[-\frac{\cos((2n+1)\pi/2-1)}{2n+1} + \frac{\cos((2n-1)\pi/2-1}{2n-1}\right] \\ &= \frac{2}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) \\ &= \frac{-4}{\pi(2n+1)(2n-1)} \end{aligned}$$

If $n = 2m-1$, the above is zero, so put $n = 2m$. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\frac{2}{2m+1} - \frac{2}{2m-1}\right] \\ &= \frac{-4}{\pi(2m+1)(2m-1)} \end{aligned}$$

Substituting into the definition of the Fourier series:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi t}{T})$$

yields the required result

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} \cos\left(\frac{2n\pi t}{T}\right) .$$

Now let a trial particular integral be

$$x = b_0 + \sum b_m \cos\left(\frac{n\pi t}{T}\right)$$

Then $b_0 = 2/\pi$. Differentiating x twice and substituting into the equation, we get

$$\left[-\left(\frac{n\pi}{T}\right)^2 + 1\right] b_m = \frac{-4}{\pi(2m+1)(2m-1)}$$

which specifies b_m.

5. The characteristics are given by

$$\begin{aligned} \frac{dx}{dt} &= 1 + y \qquad (1) \\ \frac{dy}{dt} &= y \qquad (2) \\ \frac{du}{dt} &= u + y \qquad (3) \end{aligned}$$

The boundary conditions are $x = 0, y = s, u = s(1-s)$. s is a parameter which gives a position on the boundary and t is a parameter which gives a parameter on the characteristic.

From (2) we have

$$y = s\exp(t)$$

From (1) and (2)

$$d(x-y)/dt = 1 \Rightarrow x - y = t - s$$

So

$$x = s\exp(t) + t + s$$

From (3)

$$\frac{du}{dt} = u + s\exp(t)$$

Using an integrating factor, we get

$$\frac{d(u\exp(-t))}{dt} = s$$

Hence

$$u\exp(-t) = st + C = st + s(1-s) \Rightarrow u = s\exp(t+1-s) \Rightarrow u = y(1+x-y)$$

This problem could not be solved if the boundary conditions were along $y = x$ because the characteristics cannot accomodate $y = x$.

6.

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial \xi^2} - 4\frac{\partial^2}{\partial \xi \partial \eta} + 4\frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial x \partial y} &= \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \xi \partial \eta} - 2\frac{\partial^2}{\partial \eta^2}
\end{aligned}$$

Therefore

$$4u_{xx} + 4u_{xy} + u_{yy} = 9u_{\xi\xi} + 0 + 0$$

and the equation is therefore parabolic.

Now solving for x and y, we have

$$3x = 2\xi + \eta \qquad\qquad 3y = \xi - \eta$$

Now

$$\begin{aligned}9(x^2 - xy - 2y^2) &= 4\xi^2 + 4\xi\eta + \eta^2 - (2\xi^2 - \xi\eta - \eta^2) - 2(\xi^2 - 2\xi\eta + \eta^2) \\ &= 0 + \xi\eta + 0\end{aligned}$$

so the equation is

$$\begin{aligned}9u_{\xi\xi} &= \xi\eta \\ \Rightarrow u_\xi &= \frac{1}{9}\frac{\xi^2\eta}{2} + f(\eta) \\ \Rightarrow u &= \frac{1}{54}\frac{\xi^3\eta}{2} + \xi f(\eta) + g(\eta) \\ &= \frac{1}{54}\Big((x+y)^2(x-2y) + (x+y)f(x-2y) + g(x-2y)\Big)\end{aligned}$$

This p.d.e might describe the motion of a stretched 2D membrane having inhomgeneous elastic constants. The membrane is subjected to some applied force (the RHS) the magnitude of which varies across its surface.

7. (i)

$$\begin{aligned}\mathcal{L}(H(t-a)f(t-a)) &= \int_0^\infty e^{-st}\, H(t-a)f(t-a)dt \\ &= \int_a^\infty e^{-st}\, f(t-a)dt\end{aligned}$$

Now let $\tau = t - a$ then

$$\begin{aligned}\mathcal{L}(H(t-a)f(t-a)) &= \int_0^\infty e^{-s(\tau+a)}\, f(\tau)d\tau \\ &= e^{-as}\int_0^\infty e^{-s\tau}\, f(\tau)d\tau \\ &= e^{-as}F(s)\end{aligned}$$

(ii) Taking the L.T of the equation we get

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0$$

(iii) The boundary conditions are

$$\tilde{u}(x,s) = \int_0^\infty e^{-st}u(x,t)dt$$

Thus

$$\tilde{u}(0,s) = 0$$

$$\begin{aligned}
\tilde{u}(1,s) &= \int_0^\infty te^{-st}dt \\
&= \frac{-1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\
&= 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty \\
&= \frac{1}{s^2}
\end{aligned}$$

(iv)

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0$$

Auxiliary equation is

$$m^2 - 2sm + s^2 = 0$$

Double root, solution:

$$\tilde{u} = (Ax + B)e^{sx}$$

Boundary conditions $x = 0 \Rightarrow B = 0$; $x = 1 \Rightarrow A = e^s/s^2$. Hence

$$\tilde{u} - \frac{e^s x}{s^2}e^{sx} = \frac{xe^{s(x+1)}}{s^2}$$

Therefore

$$u(x,t) = x\,(t + x + 1)\,H(t - (x+1))$$