PAPER CODE NO.
MATH724

THE UNIVERSITY
of LIVERPOOL

MAY 2001 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to five questions.

1. The equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 20\cos(t) \ ,$$

has initial conditions $y(0) = 2, y'(0) = 6$.

(i) Find the solution of this problem without using the Laplace Transform.

[8 marks]

(ii) By transforming the equation, find the value of $\tilde{y}$, the Laplace transform of y. Hence find the solution, stating explicitely each inverse Laplace transform you use.

[12 marks]

2. A function $u(x, y)$ satisfies the two dimensional Laplace equation in the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, together with the homogeneous boundary conditions:

$$u(0, y) = u(a, y) = 0, \qquad 0 < y < b$$

on $x = 0$ and $x = a$.

(i) Show that the separable solutions of this boundary value problem are

$$u_n = \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cosh\left(\frac{n\pi y}{a}\right) + D_n \sinh\left(\frac{n\pi y}{a}\right)\right]$$

where n is an integer and C_n and D_n are constants.

[10 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n given that $u(x, y)$ satisfies the boundary condition

$$u(x, 0) = 1, \ \ u(x, b) = 0, \qquad 0 < x < a$$

on $y = 0$ and $y = b$.

[10 marks]

3. Given that the Laplace transform of $f(t)$ is denoted $F(s)$, show that

(i)

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s-a}$$

[2 marks]

(ii)

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

[3 marks]

(iii)

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

[3 marks]

(iv)

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)ds$$

[5 marks]

(v) Given that $y(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-2t}$$

and the boundary conditions $y = 1$ and $\frac{dy}{dt} = -2$ at $t = 0$, show that the Laplace transform of $y(t)$, denoted $Y(s)$ satisfies

$$(s+1)(s+3)Y(s) = \frac{1}{s+2} + s + 2\ .$$

Hence find $y(t)$.

[7 marks]

4. (i) Show that for an even periodic function the Fourier coefficients b_n are zero for all n.

[5 marks]

(ii) The periodic function $f(t)$ is defined by

$$f(t) = |\sin\left(\frac{\pi t}{T}\right)| , \qquad -T < t \leq T$$

and $f(t+T) = f(t)$. Show that the Fourier series of $f(t)$ may be written

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} \cos\left(\frac{2n\pi t}{T}\right) .$$

[10 marks]

(iii) Find a particular integral of the ordinary differential equation

$$\frac{d^2x}{dt^2} + x = f(t)$$

assuming resonance does not occur.

[5 marks]

5.

The function $u(x, y)$ satisfies the first order partial differential equation

$$(1+y)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + y$$

in the domain $x > 0,\ y > 0$ and the boundary condition

$$u = y(1-y) \quad \text{on} \quad x = 0 .$$

Show that the family of characteristics of the partial differential equation may be represented by

$$x = t + se^t - s , \qquad y = se^t$$

where s and t are parameters whose significance you should explain.

[8 marks]

Hence determine the function $u(x, y)$.

[8 marks]

Explain why this problem could not be solved if the boundary conditions is along $y = x$.

[4 marks]

6. Use the transformation

$$\xi = x + y, \qquad \eta = x - 2y$$

to reduce the partial differential equation

$$4u_{xx} + 4u_{xy} + u_{yy} = 9(x^2 - xy - 2y^2)$$

to canonical form. Hence or otherwise, classify the equation.

[11 marks]

Find the general solution of this equation in terms of x and y.

[4 marks]

Suggest a physical situation in which the partial differential equation above might apply, justifying your answer

[5 marks]

7. (i) Writing $\tilde{f}(s)$ for the Laplace transform of $f(t)$ and $H(t-a)$ for the Heaviside (or unit step) function, show that the Laplace transform of $f(t-a)H(t-a)$ is

$$\tilde{f}(s)\exp(-as)\ .$$

[3 marks]

(ii) The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0\ , \qquad 0 < x < 1,\ \ t > 0\ ,$$

and the initial and the boundary conditions

$$u(x,0) = 0, \ \ \frac{\partial u}{\partial t}(x,0) = 0, \ \ \frac{\partial u}{\partial x}(x,0) = 0\ ,$$

$$u(0,t) = 0, \ \ u(1,t) = t\ .$$

Show that the Laplace transform of $u(x,t)$ with regard to t, denoted by $\tilde{u}$, satisfies the ordinary differential equation

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0\ .$$

[5 marks]

(iii) Find the boundary conditions for $\tilde{u}$ at $x = 0$ and at $x = 1$.

[2 marks]

(iv) Solve this equation for $\tilde{u}$ and hence find the function $u(x,t)$.

[5 marks]

(v) Make a sketch of the solution $u(x,t)$ for $t = 0$.

[5 marks]