MATH 701 Jan 2006

ORDINARY DIFFERENTIAL EQUATIONS

TIME ALLOWED: TWO HOURS AND A HALF

Instructions to candidates

Full marks may be obtained for complete answers to FIVE questions.
All questions will be marked, but only the five best answers counted.

1. (a) Find the general solutions for the differential equations:

$$x\frac{dy}{dx} + (1+x)y^2 = (x^3 + x^2)y^2,$$

[4 marks]

$$(1+x)\frac{dy}{dx} + 3y = x - 1,$$

[4 marks]

$$\frac{dy}{dx} = \frac{2x + 4y + 5}{x - y + 1}$$

[4 marks]

(b) Solve the initial value problem:

$$\frac{d^2y}{dx^2} + 13\frac{dy}{dx} + 40y = 40x^2 + 146x + 241, \qquad y(0) = 10, \qquad y'(0) = -31.$$

[8 marks]

2.(a) Show that $y = x$ is a solution of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0.$$

Find another linearly independent solution to this equation.

[12 marks]

(b) Show that $y = x^3$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0.$$

Find another linearly independent solution to this equation.

[8 marks]

3. Let $\phi_0(x) = 1$, $\phi_1(x) = a_0 + x$, $\phi_2(x) = b_0 + b_1 x + x^2$ and $\phi_3(x) = c_0 + c_1 x + c_2 x^2 + x^3$. Substitute each of these polynomials into the differential equation below. Show that for the equation to be satisfied by $\phi_n(x)$, $\lambda_n = n$ and find the values of the other parameters determining $\phi_n(x)$.

$$x\frac{d^2y}{dx^2} + (2-x)\frac{dy}{dx} + \lambda y = 0$$

Show that the Sturm-Liouville form of this differential equation is

$$\frac{d}{dx}\left(x^2 e^{-x}\frac{dy}{dx}\right) + \lambda x e^{-x} y = 0.$$

Suppose that $\phi_n(x)$ and $\phi_m(x)$ are nth and mth order polynomials which satisfy the above equation with λ being n and m respectively. Write this equation for the polynomial solution $\phi_n(x)$. Multiply by $\phi_m(x)$, where $n \neq m$ and integrate from 0 to ∞. Write the equation for $\phi_m(x)$, multiply by $\phi_n(x)$ and integrate from 0 to ∞. Deduce that

$$\int_0^\infty x e^{-x}\phi_n(x)\phi_m(x)dx = 0 \qquad n \neq m$$

[20 marks]

4. Show that when $\lambda \leq 4$ the boundary value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + \lambda y = 0, \qquad y(0) = 0, \qquad y'(1) = 0. \tag{1}$$

has no eigenfunctions, but that for appropriate values of $\lambda > 4$, the eigenfunctions are:

$$\phi_n(x) = e^{-2x}\sin(\omega_n x), \qquad n = 1, 2, 3\cdots,$$

where ω_n satifies the equation $\omega_n = 2\tan\omega_n$.

Write eq(1) in Sturm Liouville form and hence or otherwise, show that

$$\int_0^\pi e^{4x}\phi_n(x)\phi_m(x)dx = 0 \qquad n \neq m.$$

[20 marks]

5. Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find a series solution for the differential equation:

$$(1 - x^2)\frac{d^2y}{dx^2} + \lambda y = 0.$$

Show that the recurrence relation between the coefficients a_{n+2} and a_n is

$$\frac{a_{n+2}}{a_n} = \frac{n(n-1) - \lambda}{(n+1)(n+2)}.$$

Show that the general solution to this differential equation is a linear combination of a series of odd powers of x and a series of even powers of x

Show that if $\lambda = m(m-1)$ and m is an even positive integer, the even series solution terminates and is just a polynomial, while if m is an odd positive integer, the series of odd powers of x terminates and becomes a polynomial. Write down the polynomials for the cases when $m = 2, 3, 4, 5$. Denote these polynomials by $Q_m(x)$.

Show that for $m, n = 2, 3, 4, 5$,

$$\int_{-1}^{1} \frac{Q_n(x)Q_m(x)}{1 - x^2} dx = 0 \qquad \text{for all} \qquad m \neq n$$

[20 marks]

6. Explain what is meant by the terms *ordinary point*, *singular point* and *regular singular point* for the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials.

Find the singular points of the differential equation

$$2x(1 - x)\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

and determine whether they are regular or irregular.

Use a trial function of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

to find two linearly independent solutions of this differential equation. Write down the first three terms of each of these solutions and find their radius of convergence.

[20 marks]

7. Show that the vector $(1, 2, -1)^T$ is an eigenvector for the matrix

$$A = \begin{pmatrix} 7 & 2 & 10 \\ -8 & -3 & -16 \\ 1 & 1 & 4 \end{pmatrix}.$$

Find the eigenvalue for this eigenvector. Find the other two eigenvalues and the corresponding eigenvectors.

Find a matrix P such that

$$P^{-1}AP = D,$$

where D is a diagonal matrix whose elements should be stated.

Transform the set of differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t)$$

into the form:

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y} + \mathbf{c}(t),$$

where A is the matrix given above. Write down expressions for the components of $\mathbf{c}(t)$ in terms of the components of $\mathbf{f}(t)$.

[20 marks]

8. Show that $\mathbf{x} = (1, 3)^T e^{5t}$ is one solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \mathbf{x}.$$

Find a second solution and hence write down the general solution.

Find a linear transformation, $\mathbf{x} = P\,\mathbf{y}$, which will decouple the differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix} \mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{f}(t)$ is some known function of t and write down the decoupled differential equations. Solve these differential equations and hence determine $\mathbf{x}(t)$ when $\mathbf{f}(t) = (0, t)^T e^{5t}$.

[20 marks]