1. A Discrete Time Markov Chain with state space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ has the following transition matrix:

$$\begin{bmatrix} 0 & .3 & 0 & 0 & .7 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & .8 & 0 & 0 \\ 0 & 0 & .3 & 0 & 0 & 0 & .7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Draw the transition diagram. [2 marks]

(b) Identify closed communicating classes and the set T of transient states. Determine which closed communicating classes are periodic and which aperiodic. For the periodic classes determine the period. [2 marks]

(c) Suppose at time $n = 0$ the chain is in state 3. Find

$$\lim_{n\to\infty} P\{X_n = k\}$$

for each $k \in S$. [7 marks]

(d) Suppose at time $n = 0$ the chain is in state 2. Find the probability mass function for X_{100}, i.e. determine $P\{X_{100} = i\}$ for $i = 1, 2, \dots, 9$. [5 marks]

(e) Write down the canonical form of the transition matrix. Show how you relabelled the states. [4 marks]

2. A currency speculator makes a profit of more than £5,000 on a trading day with probability p. Let $n = 0, 1, 2, \dots$ stand for the nth trading day and X_n for the length of success run up to and including the nth day, where a profit of £5,000 or more is considered a success. For example, if the profits on the consecutive trading days starting from $n = 0$ are (in thousands of Pounds): 4.5, 7.1, -0.2, 5.7, 6.2, 9.8, 4.9 then $X_0 = 0, X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 2, X_5 = 3, X_6 = 0$.

(a) Define the state space and write down the probability transition matrix for the Markov chain $\{X_n, n \geq 0\}$. [5 marks]

(b) For each $j \geq 0$ find $\lim_{n\to\infty} P(X_n = j)$. [7 marks]

(c) Suppose the trader receives a bonus of £A for every success run of length 3 or more and a bonus of £B ($B > A$) for every success run of length 6 or more. (The length of the success run is determined after a failure occurs, so, for example, after 7 succesful trading days followed by a failure a bonus of £B is awarded and no other bonuses are in order.) Find the bonus payment rate, i.e. the average bonus per day. [8 marks]

3. Consider the Wright-Fisher genetic model: the population size N remains constant and the gene under consideration has two alleles, G and g. Assume that $2N$ genes at any generation are chosen purely at random (with replacement) from those of the previous generation.

(a) Show that X_n, number of G genes in the n-th generation, is a Discrete Time Markov Chain. Define the state space and derive the transition probabilities. [5 marks]

(b) Identify closed communicating classes and the set T of transient states. [2 marks]

(c) For a fixed initial distribution of X_0, compute $E[X_n]$ and $E[X_n(2N - X_n)]$. [10 marks]

(d) Consequently, explain mathematically the genetic drift to homozygosity. [3 marks]

4. Particles arrive at a Geiger counter according to a Poisson process with rate λ. Every time a particle arrives, the counter goes "dead" for the next $\tau > 0$ units of time.

(a) What is the probability that the counter is not dead at time t? (Consider separately the cases when $t \leq \tau$ and $t > \tau$.) [12 marks]

(b) Using a mathematical argument, show that if t is sufficiently large and τ is much smaller than $1/\lambda$, then this probability is approximately equal to $1 - \lambda\tau$. Give an intuitive explanation of this result. [8 marks]

5. Consider a two–server system in which customers arrive according to a Poisson process with rate λ. The servers are labelled as A and B. Customers going to server A require an $\exp(\mu_A)$ amount of service time, those going to server B, $\exp(\mu_B)$ amount of service time. A customer arriving at an empty system joins server A with probability α_A and server B with probability α_B ($\alpha_A + \alpha_B = 1$). Otherwise, a customer goes to the first available server. The space for waiting is infinite.

We model this system as a Continuous Time Markov Chain with state space

$$S = \{0, 1_A, 1_B, 2, 3, 4, \ldots\},$$

where 0 means no customers in the system, 2 means two customers in the system, 3 means three customers in the system etc.; 1_A means one customer in the system served by A, 1_B being defined analogously.

(a) For every state, describe the triggering events and find the transition rates. [7 marks]

(b) Write down the rate matrix and the rate diagram. [5 marks]

(c) Write down the balance equations and state the condition under which the steady-state distribution exists. [4 marks]

(d) Solve the equations for the case when $\mu_A = \mu_B = \lambda$ and α_A, α_B are arbitrary. [4 marks]

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6. (a) For a standard Brownian motion $z(t)$, prove that the correlation coefficient between $z(t)$ and $z(t+s)$ is given by

$$\rho(t, t+s) = \sqrt{\frac{t}{t+s}}, \qquad t \geq 0, \quad s \geq 0, \quad t+s \neq 0.$$

[8 marks]

(b) Compute $E[z(t_1)z(t_2)z(t_3)]$ for $t_1 < t_2 < t_3$. [12 marks]

7. Let $X_0 = 0$ and $B(t)$ be a standard Brownian motion.

(a) Solve the stochastic differential equation

$$dX = -\frac{27X}{(3X^2+1)^3}dt + \frac{3}{3X^2+1}dB.$$

[15 marks]

Hint: Consider $Y = f(X) = \frac{X^3+X}{3}$ and apply Ito's formula.
If you obtain an equation of the type $\psi(X) = B$ for some function ψ then you can leave your solution in the form $X = \psi^{-1}(B)$.

(b) Prove that $E[X(t)^3] = -E[X(t)]$ for all $t \geq 0$. [5 marks]