Full marks may be obtained for complete answers to **four** questions.

Credit will only be given for the best **four** answers.

1. An information signal (speech or music) may be represented by a weakly stationary process, $\{z_t\}(t = 0, \pm 1, \ldots)$, with mean 0, covariance function $R_z(s)$ and spectral density function $f_z(\lambda)$, that is,

$$E(z_t) = 0 \quad (\text{all } t),$$

$$\text{cov}(z_t, z_{t+s}) = R_z(s) \qquad (t, s = 0, \pm 1, \ldots),$$

$$f_z(\lambda) = (2\pi)^{-1} \sum_{s=-\infty}^{\infty} R_z(s) \exp(-is\lambda) \qquad (-\pi \le \lambda \le \pi).$$

Typically, however, z_t does not propagate well over a desired medium. A quadrature amplitude modulation system produces a transmission signal, x_t, which is in a frequency range that propagates well over the desired medium by multiplying z_t by a carrier signal, c_t, consisting of a sum of sine and cosine terms: Thus

$$x_t = z_t\, c_t,$$

$$c_t = \cos(\omega_0 t + \Theta) + \sin(\omega_0 t + \Theta),$$

where ω_0 is a fixed frequency, such that $0 \le \omega_0 \le \pi$, Θ is a uniformly distributed random variable over $[-\pi, \pi]$ and Θ and z_t are mutually independent.

Show that $\{x_t\}$ is also weakly stationary with covariance function

$$\text{cov}(x_t, x_{t+s}) = R_x(s) = R_z(s)\cos(\omega_0 s) \qquad (t, s = 0, \pm 1, \ldots).$$

[10 marks]

Show also that the spectral density function, $f_x(\lambda)$, of $\{x_t\}$ is given by

$$f_x(\lambda) = \frac{1}{2}\{f_z(\lambda - \omega_0) + f_z(\lambda + \omega_0)\}.$$

[5 marks]

The transmission signal is demodulated by the receiver by multiplying x_t by an independent carrier signal. Let

$$y_t = x_t \{\cos(\omega_0 t + \tilde{\Theta}) + \sin(\omega_o t + \tilde{\Theta})\},$$

where $\tilde{\Theta}$, Θ and z_t are mutual independent and $\tilde{\Theta}$ is also a uniformly distributed random variable over $[-\pi, \pi]$.

Question 1 continued overleaf.

Q 1 contd.

Deduce that $\{y_t\}$ is also weakly stationary and write down its covariance and spectral density functions in terms of those of the information signal, z_t.

[4 marks]

Suppose that $\{z_t\}$ is a moving average process of order 1 with spectral density function

$$f_z(\lambda) = 1 + \alpha^2 + 2\alpha \cos \lambda,$$

where $-1 < \alpha < 1$ is a known constant. Give an explicit expression for the spectral density function of $\{y_t\}$ and deduce that if $\omega_o = \frac{\pi}{2}$, $\{y_t\}$ is a purely random process. [4 marks]

Discuss why a choice of ω_0 close to 0 is usually recommended for ensuring that the original signal $\{z_t\}$ is not corrupted during the demodulation process. [2 marks]

[N.B. $\cos(A+B) = \cos A \cos B - \sin A \sin B$,
$\sin(A+B) = \sin A \cos B + \cos A \sin B$]

2. Consider a second order autoregressive process $\{x_t\}$ $(t = 0, \pm 1, \ldots)$ of the form

$$x_t = \alpha x_{t-1} + 2\alpha^2 x_{t-2} + \varepsilon_t$$

where $\{\varepsilon_t\}$ is a sequence of uncorrelated random variables, each with mean 0 and variance σ^2. For what values of α is this process stationary? [4 marks]

On the assumption that the process is stationary, show how to express x_t in the form

$$x_t = \sum_{j=0}^{\infty} b(j)\,\varepsilon_{t-j}$$

and obtain the coefficients $b(j)$ explicitly in terms of α. [6 marks]

Derive a recurrence relation for the autocorrelation function, $r(s)$, of $\{x_t\}$ and show that

$$r(1) = \frac{\alpha}{1-2\alpha^2}, \qquad r(2) = 2\alpha^2 + \left\{\frac{\alpha^2}{1-2\alpha^2}\right\}.$$ [7 marks]

Hence, or otherwise, show that the process variance, $R(0)$, may be written in terms of α and σ^2 as follows:

$$R(0) = \frac{\sigma^2\left(1-2\alpha^2\right)}{1-3\alpha^2-6\alpha^4+8\alpha^6}.$$ [4 marks]

In a realization of T consecutive observations from $\{x_t\}$, the estimated values of the sample correlation function, $r^{(T)}(s)$, for $s = 1$, 2 and 3 are as follows

s	1	2	3
$r^{(T)}(s)$	0.5	0.46	0.2

Determine the corresponding Yule-Walker estimate of α. [4 marks]

3. Let $\{x_t\}$ $(t = 0, \pm 1, \ldots)$ be a (weakly) stationary process with zero mean and an absolutely summable covariance function, $R_x(u)$. Consider a new process, y_t, obtained from x_t by a linear filtering operation of the form

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j}, \qquad (t = 0, \pm 1, \ldots),$$

where $$\sum_{j=-\infty}^{\infty} |b_j| < \infty.$$

Show that $\{y_t\}$ is also weakly stationary and the covariance function, $R_y(u)$, and the spectral density function, $f_y(\lambda)$, of y_t are given by

$$R_y(u) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_j b_k \; R_x(u - k + j),$$

and $$f_y(\lambda) = |B(\lambda)|^2 f_x(\lambda),$$

where $$B(\lambda) = \sum_{j=-\infty}^{\infty} b_j e^{-ij\lambda},$$ [6 marks]

denotes the transfer function of the filtering operation. Suppose that $\{x_t\}$ is a sequence of uncorrelated random variables each with zero mean and unit variance and y_t is the pth difference of x_t, i.e.

$$y_t = \Delta^p x_t,$$

where $p \geq 1$ is an integer and $\Delta x_t = x_t - x_{t-1} = (1 - z)x_t$, with $z^j x_t = x_{t-j}$.

Show that the covariance function of y_t is given by

$$\begin{aligned} R_y(u) &= (-1)^u \binom{2p}{p-u}, \quad && u = 0, 1, \ldots, p, \\ &= 0 && u > p. \end{aligned}$$

If $r_y(1) = R_y(1)/R_y(0)$ then verify that as $p \to \infty, r_y(1) \to -1$. [8 marks]

Question 3 continued overleaf

Q 3 contd.

The p first difference operations are followed by h-iterated summation operations of the form

$$w_t = (1+z)^h y_t .$$

Show that the spectral density function of w_t is

$$f_w(\lambda) = \frac{1}{2\pi} 2^h 2^p (1+\cos\lambda)^h (1-\cos\lambda)^p ,$$ [4 marks]

and determine the frequency $\lambda \in (0,\pi)$ at which $f_w(\lambda)$ peaks. Sketch $f_w(\lambda)$ and hence compare the behaviour of w_t and x_t. [7 marks]

[Hint: For any positive integers m, n and $s \le \min(m, n)$

$$\sum_{x=0}^{s} \binom{n}{x}\binom{m}{s-x} = \binom{m+n}{s}$$]

4 a) In a study of the effects of seat belt legislation on British road casualties, A.C. Harvey and J. Durbin consider a class of "structural" models. A basic structural model for a non-seasonal time series is as follows:

$$y_t = \mu_t + \varepsilon_t$$

$$\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t$$

$$\beta_t = \beta_{t-1} + \xi_t$$

where $\{\varepsilon_t\}$, $\{\eta_t\}$ and $\{\xi_t\}$ are three independent sequences of independent normal variables each with mean 0 and variance given by

$$V(\varepsilon_t) = \sigma^2,\ V(\eta_t) = \tau^2,\ V(\xi_t) = \theta^2.$$

Show that the process $\{y_t\}$ postulated in this model can be made stationary by taking second differences and derive the correlation function of the resulting stationary process, $\{z_t\}$ say.

[10 marks]

Indicate whether an ARMA (p, q) model may be found with the same correlation structure as that of $\{z_t\}$ and, if so, give the relevant values of p and q. [2 marks]

[N.B. You are not required to express the parameters of such an ARMA model in terms of those of $\{z_t\}$.]

b) An alternative to the basic "structural" model described in a) above is the linear trend plus error model.

$$y_t = a + bt + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a sequence of independent normal variates, each with mean 0 and variance σ^2. Show that taking second differences of y_t eliminates the trend, leaving a stationary stochastic process, with a mean of zero. Obtain the correlation function of this stationary process and discuss whether it is also invertible. [10 marks]

c) Explain the relationship between the models in a) and b) above. Are there circumstances in which they are equivalent? [3 marks]

5. Let $\{x_t\}(t = 0, \pm 1, \ldots)$ be a stationary process with mean 0, an absolutely summable covariance function, *R(u)*, and a nonvanishing spectral density function, $f(\lambda)$. The inverse covariance function of $\{x_t\}$ is defined by

$$Ri(u) = \int_{-\pi}^{\pi} e^{iu\lambda} \; fi(\lambda)\, d\lambda \qquad (u = 0, \pm 1, \ldots),$$

and the inverse correlation function by

$$ri(u) = Ri(u) / Ri(0),$$

where $fi(\lambda) = (2\pi)^{-2} \{f(\lambda)\}^{-1}$ denotes the inverse spectral density function.

Suppose that $\{x_t\}$ is an autoregressive process of order 1,

$$x_t = \alpha x_{t-1} + \varepsilon_t, \quad |\alpha| < 1, \qquad (*)$$

where $\{\varepsilon_t\}$ is a sequence of independent Normal variables, each with mean 0 and variance σ^2 and the spectral density function of $\{x_t\}$ is given by

$$f(\lambda) = (\sigma^2 / 2\pi) |1 - \alpha \exp(-i\lambda)|^{-2}.$$

Show that the inverse correlation function of $\{x_t\}$ is given by

$$ri(1) = -\alpha / (1 + \alpha^2) = ri(-1),$$

$$ri(u) = 0, \qquad |u| > 1.$$

[7 marks]

Given a realization of T consecutive observations, $x_1, \ldots, x_T$, from the 1st order autoregressive process $(*)$, write down the likelihood function of the observations. [12 marks]

Suppose now that for some $1 < s < T$, x_s has not been observed and it is treated as a fixed missing value. Show that the maximum likelihood estimator of x_s when α is treated as known is given by

$$\hat{x}_s = -ri(1)\{x_{s-1} + x_{s+1}\}.$$

[4 marks]

Discuss possible implications of this result for the estimation of x_s when α is unknown.

[2 marks]

6. Let $\{x_t, t=1,...,T\}$, where T is even, be a sequence of independent normal random variables, each with mean 0 and variance σ^2. Put $n = \frac{T}{2}$,

$$\lambda_j = 2\pi j / T \qquad (j = 0, 1, ..., n),$$

$$a_T(\lambda_j) = \sum_{t=1}^{T} x_t \cos(t\lambda_j), \qquad b_T(\lambda_j) = \sum_{t=1}^{T} x_t \sin(t\lambda_j)$$

and let

$$I_T(\lambda_j) = (2\pi T)^{-1} \left| \sum_{t=1}^{T} x_t \exp(-it\lambda_j) \right|^2 \qquad (j = 0, 1, ..., n)$$

be the periodogram function. Show that

$$\{a_T(\lambda_0)\, a_T(\lambda_1), ..., a_T(\lambda_n);\ b_T(\lambda_1), ..., b_T(\lambda_{n-1})\}$$

are independent normal random variables, each with mean 0 and variances given by

$$V(a_T(\lambda_0)) = V(a_T(\lambda_n)) = T\sigma^2,$$

$$V(a_T(\lambda_j)) = V(b_T(\lambda_j)) = \frac{T}{2}\sigma^2 \qquad (j = 1, ..., n-1).$$

[10 marks]

Hence find the joint distribution of the random variables

$$\{I_T(\lambda_0), I_T(\lambda_1), ..., I_T(\lambda_n)\}.$$ [5 marks]

Explain succinctly why the periodogram function does not provide a satisfactory estimator of the spectral density function of a stationary process. [5 marks]

Indicate briefly how an improved estimator may be obtained. [5 marks]

[N.B. 1) You may assume without proof the following results:

a) $\sum_{t=1}^{T} \cos\frac{2\pi jt}{T} \cos\frac{2\pi kt}{T} = 0 \qquad j \neq k$

$= n \qquad j = k \neq 0, n$

$= T \qquad j = k = 0, n$

Question 6 continued overleaf

Q 6 contd.

b) $$\sum_{t=1}^{T} \sin\frac{2\pi jt}{T}\sin\frac{2\pi kt}{T} = n \qquad j = k \neq 0, n$$

$$= 0 \qquad \text{otherwise}$$

c) $$\sum_{t=1}^{T} \sin\frac{2\pi jt}{T}\cos\frac{2\pi kt}{T} = 0 \qquad \text{all } j, k$$

2) If χ_k^2 denotes a chi-squared random variable with *k* degrees of freedom,

$E\left(\chi_k^2\right) = k, \quad V\left(\chi_k^2\right) = 2k.$]

7. Explain what is meant by *stylised features of a financial time series*. Briefly describe three such features as applicable to changes in share prices. [4 marks]

Consider the stochastic variance model:

$$y_t = \sigma_t \varepsilon_t$$

where y_t denotes the observed time series, ε_t is a sequence of independent normal random variables each with mean 0 and variance 1, and

$$\sigma_t^2 = \exp(h_t),$$

with

$$(h_t - \mu) = \alpha(h_{t-1} - \mu) + \eta_t, \qquad |a| < 1.$$

Here $\{\eta_t\}$, independent of $\{\varepsilon_t\}$, is a sequence of independent normal random variables each with mean 0 and variance τ^2, and μ and α are known constants such that $\mu = E(h_t)$ and $|\alpha| < 1$.

Show that $\{y_t\}$ is a sequence of uncorrelated random variables and write down an expression for the variance of y_t. [5 marks]

Let

$$K(y) = E(y_t^4) / \{E(y_t^2)\}^2$$

be a measure of kurtosis of y_t, and define $K(\varepsilon)$ analogously but with ε_t replacing y_t. Show that

$$K(y) > K(\varepsilon),$$

that is, y_t has a higher kurtosis than ε_t. [4 marks]

Derive the autocorrelation function of $z_t = \log y_t^2$. [10 marks]

Compare and contrast the correlation functions of $\{y_t\}$ and $\{z_t\}$ and comment on the potential usefulness of the stochastic variance model for analysis of financial time series. [2 marks]

[N.B. You may use without proof the following results

i) that $\{h_t\}$ and $\{\varepsilon_t\}$ are independently distributed and $\{\varepsilon_t\}$ is also independent of all continuous functions of $\{h_t\}$;

Question 7 continued overleaf

Q 7 contd.

ii) that $\sigma_t^2 = \exp(h_t)$ has a log-normal distribution and its *k*th moment about the origin is given by

$$E\left\{\left(\sigma_t^2\right)^k\right\} = \exp\left\{k\mu + \frac{1}{2}k^2\phi^2\right\},$$

where

$$\phi^2 = \mathrm{var}(h_t) = E\left[\{h_t - \mu\}^2\right];$$

iii) if χ_1^2 denotes a chi-squared random variable with one degree of freedom

$$E\left(\log \chi_1^2\right) = -\gamma - ln 2 = -1.3, \quad \mathrm{var}\left(\log \chi_1^2\right) = \frac{\pi^2}{2},$$

where γ= 0.577215 is Euler's constant;

iv) $\mathrm{cov}(h_t, h_{t+u}) = \dfrac{\alpha^{|u|}\,\tau^2}{1-\alpha^2} \qquad (t, u = 0, \pm 1, \ldots).$]