1. Computer A sends a message to computer B over an unreliable telephone line. The message is encoded so that B can detect when errors have been introduced into the message during transmission. If B detects an error, it requests A to retransmit the message. The probability of a message transmission error is $q = 0.1$ for each transmission, independently of all other transmissions.

(a) What is the probability that the message needs to be transmitted (i) more than once, (ii) more than twice, (iii) more than k times, $k = 1, 2, \ldots$? [4 marks]

Write down the probability mass function for random variable X=number of transmissions till the message is received without errors. [2 marks]

What is the mathematical expectation of X ? [3 marks]

(b) Suppose instead that computer A sends two identical copies of each message to computer B simultaneously over two unreliable telephone lines working independently. Computer B can detect when errors have occured in either line. Let the probability of message transmission error in line 1 and line 2 be q_1 and q_2 repspectively. Computer B requests retransmissions until it receives an error-free message on either line. Find the probability that more than k transmissions are required for a given message. [5 marks]

What is the probability that, given that the last transmission was free of errors, the error-free message was received on line 2 and not on line 1? [6 marks]

2. A store has to supply items, and the demand during time period $[0, t]$ follows the Poisson distribution with paramter $m = \lambda t$.

(a) What is the distribution of time T until the first request? [1 mark]
Let $E[T] = 7.5$ days and consider a one month period (i.e. $t = 30$ days). Determine the values of λ and m to be used below. [2 marks]

(b) Suppose the stock is made up to 6 articles at the beginning of a month.

(i) Calculate the expected number of the articles remaining in the store at the end of the month. [4 marks]

(ii) Calculate the probability that the demand will be satisfied. [4 marks]

(iii) Suppose the demand is satisfied. What is the expected number of the articles remaining in the store at the end of the month? [4 marks]

(c) How many articles must be kept in stock at the beginning of the month to be 97% sure of satisfying the demand during the month? [5 marks]

3. (a) Justify the Box-Muller method of generating normal random variables. That is

$$Z_1 = (-2 \ \ln U_1)^{1/2} \cos(2\pi U_2) \quad \text{and} \quad Z_2 = (-2 \ \ln U_1)^{1/2} \sin(2\pi U_2)$$

are standard normal and independent random variables, provided U_1 and U_2 are two independent standard uniform random variables. [15 marks]

(b) Let $V(t)$, $t \geq 0$ be a random process with the increments $\Delta_s = V(t+s) - V(s)$ over non-overlapping intervals being independent normal random variables $N(0, s)$ with zero mean and $Var = s$. (Such a process is called 'Brownian motion'.) One is interested in the values $V(1), V(2), V(3)$ and $V(4)$ given that $V(0) = 0$.

Suppose the uniform random generator has produced numbers

U_1	0.6540	0.3104
U_2	0.4638	0.4316

Simulate the values $V(1), V(2), V(3)$ and $V(4)$ using the Box-Muller method. [5 marks]

4. Let X be a non-negative absolutely continuous random variable with hazard function $h(t)$. Write down the expression for cumulative distribution function $F(t)$ in terms of $h(\cdot)$. [4 marks]

Assume that the hazard function, $h(t)$, of the length of human life, W, say, increases almost exponentially once a person reaches 25 years of age, but that $h(t)$ starts at a very low value. A plausible probability model for this situation specifies that

$$h(t) = a\ e^{bt},$$

where $a > 0$ and $b > 0$ are parameters such that a is very small.

(a) Find the cumulative distribution function, $F_W(t)$, and the probability density function, $f_W(t)$, of W. [5 marks]

(b) Suppose that the length, W, of a man's life in a certain community follows the hazard function $h(t)$, given above with $b = \ln(1.1)$, i.e.

$$h(t) = a(1.1)^t$$

and the rate of mortality increases by 10% each year, where $a > 0$ is a constant. It is known also that the probability that a man in reasonable good health at age 52 dies before he reaches the age of 53 is 1%, that is,

$$P(W \leq 53 | W > 52) = 0.01.$$

Find the corresponding value of a. [8 marks]

(c) For the values of a and b indicated above estimate the probability that $W > 200$. [3 marks]

5. At the beginning of each month t, the manager of a warehouse determines the current inventory of a product $s_t \in \{0, 1, 2, 3\}$ and decides how much to order from the supplier. The total level of inventory cannot exceed 3 items. Demand for the product, D_t, arrives throughout the month but all orders are filled on the last day of the month. Random variables D_t are independent and have the same Probability Mass Function:

$$P(D = 0) = \frac{1}{4}; \quad P(D = 1) = \frac{1}{2}; \quad P(D = 2) = \frac{1}{4}.$$

If the manager orders $a_t = a > 0$ units the corresponding cost is $O(a) = 4+2a$, and $O(0) = 0$. The cost of maintaining an inventory of u units for a month (between receipt of the order and releasing inventory to meet demand) equals $h(u) = u$. Finally, if the demand is j units and sufficient inventory is available, the manager receives a revenue of $8j$. There is no backlogging of unfilled orders, so that excess demand is lost.

(a) Calculate $F(u)$, the expected value of the revenue in month t when the inventory at the beginning of the month (i.e. after the supplier performs the contract and prior to receipt of customer orders) was u. [3 marks]

(b) Formulate the problem as a Markov Decision Process, i.e. identify the state and action spaces S and $A(s)$, transition probabilities $p(j|s, a)$, and one step reward $r(s, a)$. [4 marks]

(c) Write down the Bellman equation in terms of s, a, p, r. [4 marks]

(d) Calculate the optimal ordering policy for a two months period and the corresponding maximal expected revenues for all possible initial states. [9 marks]

6. (Parrando's paradox.) A counter performs an irreducible random walk on the vertices 0,1,2 of the triangle in the figure beneath, with transition matrix

$$P = \begin{bmatrix} 0 & p_0 & q_0 \\ q_1 & 0 & p_1 \\ p_2 & q_2 & 0 \end{bmatrix}$$

where $p_i + q_i = 1$ for all i.

Clockwise means:

(a) Determine the stationary distribution π (i.e. limiting probabilities π_i). [5 marks]

(b) Suppose you gain one pound for each clockwise step of the walk, and you lose one pound for each anticlockwise step. Compute the mean yield γ per step in equilibrium (i.e. the long-run expected cost or return per unit time). [4 marks]

(c) Suppose there are two actions in each state: if $a = 1$ then $p_0 = p_1 = p_2 = 0.49$; if $a = 2$ then $p_0 = 0.09$ and $p_1 = p_2 = 0.74$.
(i) What is γ_1, the expected return per unit time, if the counter applies the strategy: set $a = 1$ always? [3 marks]
(ii) What is γ_2, the expected return per unit time, if the counter applies the strategy: set $a = 2$ always? [3 marks]

(d) Suppose at each step the counter is equally likely to choose $a = 1$ and $a = 2$. What are p_i in this case? [1 mark]
Compute the yield γ_3 for this strategy. [3 marks]

(e) Which of the strategies in (c) and (d) are unprofitable and which are profitable? [1 mark]