1. Rod and Fred are playing a game of tennis, and the game stands at deuce. In this situation, Rod wins the game as soon as he has won two more points in total than Fred. Fred can win the game in the same way. Rod wins any point with probability p, independently of any other point.

(a) If $\gamma = P(\text{Rod wins the game})$ given that the game stands at deuce show that

$$\gamma = \frac{p^2}{1-2p(1-p)}.$$

[9 marks]

Hint. You can consider all four possible outcomes of the next two services and use the law of total probability to produce an equation with respect to the unknown γ.

(b) If Fred trained the day before then $p = 0.49$, if not then $p = 0.51$. Rod does not know exactly whether Fred trained or not, but probability $P(\text{Fred trained})$ is 0.75. What is the probability $\gamma = P(\text{Rod wins the game})$ given that the game stands at deuce in this situation? [5 marks]

(c) Suppose Rod won the game (after it stood at deuce). What is the probability that Fred had trained the day before? [5 marks]

Is this probability less than the unconditional probability that Fred had trained the day before? Give your comments in this connection. [1 marks]

2. A signal, which can take values $s = 0$ or $s = 1$ is to be transmitted from computer A to computer B subject to channel noise disturbance. The value received at B, is given by

$$R = s + Z,$$

where Z is uniformly distributed on the segment $[-0.6 \ \ +0.6]$. If $R > 0.6$ then $s = 1$ and if $R < 0.4$ then $s = 0$, but if $R \in [0.4, \ \ 0.6]$ then computer B cannot decode the message received, and in this case it requests A to retransmit the signal. Assume that noise Z affects different signals independently of each other.

(i) Compute the probabilities that the signal needs to be transmitted more than once if $s = 0$ and if $s = 1$ and show that these probabilities coincide.

[6 marks]

(ii) Show that the probability mass function for RV X=*number of transmissions till the signal is properly decoded* is given by

$$P(X = k) = p^{k-1} \times (1 - p), \qquad k = 1, 2, \ldots$$

where p is the common probability computed in (i) above. (This distribution is called geometric.) Calculate the mathematical expectation of X. [4 marks]

(*Hint*: If $\alpha \in (-1, 1)$ then

$$\frac{1}{(1-\alpha)^2} = \sum_{i=1}^{\infty} i\alpha^{i-1}.)$$

What is the expected number of messages sent by both computers per one correctly decoded signal? (You should count the messages from B asking for retransmission as well as the signals from computer A.) [4 marks]

(iii) Since the round-trip time between computers A and B is large it was decided to transmit each signal twice, without waiting for the reply from B. Thus, retransmission is needed only if the both signals sent cannot be decoded. What is the expected number of messages sent by either computer per one correctly decoded signal in this situation? [6 marks]

3. A farmer sells seeds of beautiful aster, but not all seeds are good.

(i) Ten people bought 100 seeds each and found the following amounts of dead seeds:

$$x_i : \quad 28, \quad 22, \quad 31, \quad 31, \quad 25, \quad 35, \quad 35, \quad 30, \quad 31, \quad 31.$$

Compute estimators of the mean ($\hat{\mu}$) and of the variance ($\hat{\sigma}^2$) of RV

$$X = \text{"number of dead seeds in a packet of 100."}$$

[2 marks]

Assume that each seed is dead with probability p independent of others. Estimate p using $\hat{\mu}$ [1 marks]

Suggest a mathematical model for X using the Central Limit Theorem. [3 marks]

Having the estimator $\hat{p}$ obtained compute the estimator of $Var(X)$ using your model and compare it with $\hat{\sigma}^2$ found earlier. [2 marks]

(ii) Evaluate probabilities $P(X \leq 20)$ and $P(X \geq 37)$ using the model suggested and the estimator $\hat{p}$. [4 marks]

(iii) Several years later, this experiment was repeated with the following results

$$y_i : \quad 26, \quad 20, \quad 33, \quad 33, \quad 23, \quad 37, \quad 37, \quad 28, \quad 33, \quad 29.$$

Answer all questions in (i) for this data set. [6 marks]

Comment on whether this data set may be regarded as a random sample from the same distribution as that of X suggested above. [2 marks]

4. (a) You have observations $x_1, x_2, \ldots, x_n$ of life times of a certain device. Assuming that this lifetime is exponentially distributed, $F(x) = 1 - e^{-\lambda x}, \;\; x \geq 0$ write down the likelihood function and derive the maximum likelihood estimator $\hat{\lambda}$ for the unknown parameter λ. [7 marks]

(b) Observed data ($n = 35$) are presented in the table

No.	Interval (in years)	Frequency
1	0 – 0.5	12
2	0.5 –1.5	8
3	1.5 – 2.5	7
4	2.5 – 5.5	8

Compute the maximum likelihood estimator $\hat{\lambda}$ assuming that all the observations are concentrated at the middle of the interval No. 2,3, and 4 (e.g. there are 8 observations of 1 year, 7 observations of 2 years and so on) and all 12 observations from the first interval equal 0. [2 marks]

(c) Compute the sample cumulative distribution and plot it with appropriate scale on the vertical axis. [6 marks]

Does the experimental model fit the data? [2 marks]

Using your graph, estimate parameter λ and compare it with $\hat{\lambda}$ obtained earlier. [3 marks]

5. (a) A random variable X is distributed as Pareto with probability density function

$$f(x) = \begin{cases} (\delta/\theta)\left(1+\frac{x}{\theta}\right)^{-\delta-1}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where $\delta > 1$ and $\theta > 0$ are the parameters of the distribution. Determine the cumulative distribution function of X and $E[X]$. [6 marks]

(b) The lifetimes X_i $(i = 1, 2, 3, 4)$ of components C_i are independently distributed, each being Pareto with probability density function $f(x)$ defined above. A piece of equipment has been assembled by placing two of these components in a sequence and using the schematic design shown below:

The piece will work only if both components C_1 and C_2 are working. Let Y denote the lifetime of the piece. Determine the probability density function of Y and $E[Y]$. [5 marks]

Derive the hazard function of this piece and discuss its meaning. [2 marks]

(c) Suppose instead that a second piece of equipment has been designed by placing two of these components in parallel and using the scheme shown below:

This second piece will fail only if both the components C_3 and C_4 have failed. Let W denote the lifetime of this second piece. Determine the probability density function of W and $E[W]$. [6 marks]

(d) Discuss briefly the advantage of the parallel placing of components, from the point of view of reliability. [1 marks]

6. Most of digital pseudo-random generators produce sequences of numbers according to the recursive relation

$$X_{i+1} = g(X_i), \quad i = 0, 1, 2, \ldots$$

so that $\forall i$ $X_i \in [0,1]$ exhibits properties of a standard uniform RV U(0,1).

(i) Give the definition of the period of a generator. [1 marks]

(ii) Suppose CDF $F(y)$ is absolutely continuous and strictly increasing and hence there exists the inverse function $F^{-1}(\cdot)$. If $x \sim U(0,1)$ has been generated, how can you compute a new RV y having the given CDF $F(y)$? [1 marks]

(iii) Suppose the following two values of the RV uniform on [0,1] have been generated:

$$x_i : \quad 0.2311; \quad 0.6068.$$

Using this data, generate two values, y_i, of the Cauchy RV having density

$$f(y) = \frac{1}{\pi(1+y^2)}, \qquad -\infty < y < +\infty.$$

[5 marks]

(iv) Having the following 20 realizations of the Cauchy RV y_i

-1.126	0.349	-0.044	2.814	1.079	-0.138	-17.187
1.591	-0.175	0.379	1.305	3.988	0.928	-1.617
-0.305	4.867	3.743	-0.289	2.879	-5.437	

compute the average values

$$\bar{y}_1 = y_1, \; \bar{y}_2 = \frac{1}{2}(y_1 + y_2), \; \bar{y}_3 = \frac{1}{3}(y_1 + y_2 + y_3), \; \ldots, \; \bar{y}_{20} = \frac{1}{20}\sum_{i=1}^{20} y_i.$$

Is there any evidence that $\bar{y}_n$ stabilizes as n increases? Comment. [4 marks]

(v) Using the same initial values

$$x_i : \quad 0.2311; \quad 0.6068$$

generate two values, z_i, of the exponential RV having density $f(z) = e^{-z}$, $z \geq 0$. [5 marks]

Having the following 20 realizations of the exponential RV z_i

0.263	0.933	0.666	2.219	1.436	0.609	0.019
1.723	0.588	0.956	1.569	2.548	1.340	0.194
0.520	2.741	2.477	0.528	2.241	0.059	

compute the realized average values

$$\bar{z}_1 = z_1, \; \bar{z}_2 = \frac{1}{2}(z_1 + z_2), \; \bar{z}_3 = \frac{1}{3}(z_1 + z_2 + z_3), \; \ldots, \; \bar{z}_{20} = \frac{1}{20}\sum_{i=1}^{20} z_i$$

and discuss the stabilization properties of $\bar{z}_n$ as n increases. [4 marks]

7. In the Goel-Okumoto model, the number of software failures in interval $[t_1, t_2)$ during the debugging phase is the Poisson RV $X(t_1, t_2)$:

$$P(X(t_1, t_2) = k) = \exp[-\mu(t_2) + \mu(t_1)] \frac{[\mu(t_2) - \mu(t_1)]^k}{k!},$$

where $\mu(t)$ is a given increasing function, $\mu(0) = 0$, and $X(t_1, t_2)$ and $X(t_3, t_4)$ are independent for non-overlapping intervals $[t_1, t_2)$ and $[t_3, t_4)$.

Let

$$\lambda(t) = \frac{d\mu(t)}{dt} = at^{b-1}, \quad 0 < b < 1, \qquad a > 0$$

be the failure intensity for this model.

(i) Suppose time t is measured in hours and $a = 1$ and $b = \frac{1}{2}$. Determine the minimal integer $T > 1$ such that the probability of two or more failures during the interval $[1, T)$ is greater than $\frac{1}{2}$. [6 marks]

(ii) Again, let $a = 1$ and $b = \frac{1}{2}$. Suppose two specialists work consecutively on this software, one hour each. Let $X(0, 1)$ denote the number of failures experienced by the first specialist and $X(1, 2)$ that by the second specialist. Find $E[X(0, 1)]$ and $E[X(1, 2)]$ and deduce that

$$E[X(0, 1)] > 2E[X(1, 2)].$$

Comment on plausibility of the proposed model for assessing software reliability. [2 marks]

Suppose now that the manager pays £30 to each specialist if he faces a failure (and corrects corresponding error in the program), but after two failures the manager stops this process and returns software to a special division. Determine the expected gain of each specialist while working on this software? [6 marks]

(iii) Suppose now that

$$\lambda(t) = ce^{-dt}, \qquad c > 0, \ d > 0$$

and compute $E_2 = E[X(0, T)]$ for this "second" model. [4 marks]

Write down the similar mathematical expectation $E_1 = E[X(0, T)]$ for the initial model with

$$\lambda(t) = at^{b-1}, \quad 0 < b < 1, \qquad a > 0$$

and compare these mathematical expectations for large T (i.e. compare the limits $\lim_{T \to \infty} E_i$). Give your comments on the adequacy of the two models considered. [2 marks]