(Summer 2006)

Time allowed: Three Hours.

You may attempt as many problems as you like. The best FOUR answers will be taken into account. Each question carries the same weight.

1. [25 marks]

(i) Explain what it means for $\omega = -1/2 + \sqrt{-3}/2$ to be a primitive 3rd root of unity. Find the minimal equation of ω over $\mathbb{Q}$. [5 marks]

(ii) Write down formulae which relate the roots of quadratic and cubic equations to their coefficients, and explain how one can obtain them. [5 marks]

(iii) Denoting the roots by $\alpha = y + z$, $\beta = \omega y + \omega^2 z$, $\gamma = \omega^2 y + \omega z$, solve the cubic equation

$$x^3 + 9x - 4 = 0$$

in radicals. [15 marks]

2. [25 marks]

(i) Denote the roots of the quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0 \quad (1)$$

by α_1, α_2, α_3 and α_4. Define the elementary *symmetric functions* of the variables α_1, α_2, α_3 and α_4, and write down the relation between the roots and coefficients of the equation (1) using the elementary symmetric functions. [5 marks]

(ii) Formulate the Theorem about symmetric functions of the variables α_1, α_2, α_3 and α_4, and illustrate it by deriving the formula for $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$ in terms of the elementary symmetric functions. [5 marks]

(iii) By considering the polynomials

$$U = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4),\ V = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4),\ W = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)$$

of the roots, explain how to reduce the solution of the quartic equation (1) to a cubic equation. [12 marks]

(iv) Find one of the coefficients of the cubic equation. [3 marks]

3. [25 marks]

(i) Factorize the polynomial $x^5 - 2x^3 - 3x^2 + 6$ into irreducibles over $\mathbb{Q}$. [8 marks]

(ii) Prove that the polynomial $x^5 + x^3 + x^2 + x + 1$ is irreducible over the field $\mathbb{F}_2$ with two elements. [6 marks]

(iii) Use (ii) to construct an infinite series of irreducible polynomials of degree 5 over $\mathbb{Q}$. Formulate exactly the statement (by Gauss) which you have to use. [6 marks]

(iv) Use (ii) to construct a field with 32 elements. [5 marks]

4. [25 marks]

(i) Explain what is meant by saying that a point in the Euclidean plane is constructible by ruler and compass from a given set of points. [3 marks]

(ii) State without proof a necessary condition for (i) to hold in terms of the degree of a certain field extension. [3 marks]

(iii) State without proof a sufficient and necessary condition for (i) to hold, using Galois Theory. [4 marks]

(iv) Establish the formula

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

[Hint: Take the real part of $(\cos x + i\sin x)^5$.]

[5 marks]

(v) By choosing (for example) an angle θ such that $\cos 5\theta = 5/7$, deduce that in general an angle cannot be divided into 5 equal parts using only ruler and compass.

[10 marks]

5. [25 marks]

(i) Define the splitting field K and the Galois group $G = G(P)$ of a polynomial $P = P(x)$ over a field k. [4 marks]

(ii) When does the Main Galois Theorem hold for the field extension $k \subset K$ in (i)? [4 marks]

(iii) State the Main Galois Theorem for the field extension $k \subset K$ in (i) under the condition (ii), including the formulae for the degrees of the corresponding field extensions in terms of orders of subgroups. [7 marks]

(iv) Determine the splitting field K and the Galois group G for the polynomial $x^3 - 5$ over $\mathbb{Q}$. List all subgroups of G, and use the Main Galois Theorem to describe all fields between K and $\mathbb{Q}$. [10 marks]

6. [25 marks]

(i) Write down all possible cycle types of permutations in the alternating group A_5. [4 marks]

(ii) Express the following permutations of the alternating group A_5 as products of disjoint cycles:

$$(ijk)(klm), \quad (ij)(lm)(jk)(lm), \quad (ijk)(jkl).$$

[Here i, j, k, l, m are all different.]

Deduce that A_5 can be generated by elements of order 2 only, and can also be generated by elements of order 3 only. [5 marks]

(iii) Deduce that there does not exist a homomorphism of A_5 onto any nontrivial cyclic group of prime order, and hence prove that A_5 is not soluble. [4 marks]

(iv) For each integer $m \geq 2$ prove that the polynomial

$$P(x) = x^5 - 5mx + 5$$

over $\mathbb{Q}$ has Galois group which is isomorphic to S_5. Hence show that the equation $P(x) = 0$ is not soluble in radicals. [You can use results from the course, provided that you state them clearly.] [12 marks]