Time allowed: 3 hours.

You may attempt as many problems as you like. The best FOUR answers will be taken into account. Each question carries the same weight.

1. [25 marks]

For the real matrix

$$J = \begin{pmatrix} E_m & 0 \\ 0 & -E_m \end{pmatrix}$$

where E_m is the $m \times m$ identity matrix and 0 is the $m \times m$ zero matrix consider the set of matrices (the real orthogonal group of signature (m, m))

$$O(m, m) = \{X \in Mat_{2m}(\mathbb{R}) \mid X^t J X = J\}$$

where X^t is the transpose matrix to X.

(a) Prove that $O(m, m) \subset GL(2m, \mathbb{R})$ is a subgroup. [5 marks]

(b) For the Cayley transform $X = (E - Y)(E + Y)^{-1}$, where $Y \in Mat_{2m}(\mathbb{R})$ and Y is near 0, show that the matrix X is in $O(m, m)$ if and only if

$$Y^t J = -JY.$$

[10 marks]

(c) Prove that $O(m, m)$ is a Lie group and that $\dim O(m, m) = 2m^2 - m$.
[10 marks]

2. [25 marks]

Let V be an n-dimensional vector space over $\mathbb{R}$.

For $0 \leq m_1 \leq m_2 \leq m_3 \leq n$, consider the flag variety

$$Fl(n; m_1, m_2, m_3) =$$
$$\{W_1 \subset W_2 \subset W_3 \subset V \mid \dim W_1 = m_1, \dim W_2 = m_2, \dim W_3 = m_3\},$$

where W_1, W_2 and W_3 are vector subspaces of V.

(a) By considering the action of the group $GL(V)$ on V, prove that $Fl(n; m_1, m_2, m_3)$ is a manifold. Find the dimension of $Fl(n; m_1, m_2, m_3)$.
[15 marks]

(b) By considering the action of the orthogonal group $O(n)$, prove that $Fl(n; m_1, m_2, m_3)$ is compact. [7 marks]

(c) Use (a) to find the dimension of the space

$$\{\text{line} \subset \text{plane} \subset \text{3-dimensional space} \subset \text{4-dimensional affine space}\}$$

of affine subspaces over $\mathbb{R}$. [3 marks]

3. [25 marks]

(i) Give definitions of the following Lie groups and find their connected components:

(a) $GL(n, \mathbb{R})$; [12 marks]

(b) $O(n)$ (you need only outline changes which are required in comparison with the previous case); [9 marks]

(ii) Prove that the flag variety $Fl(n; m_1, m_2, m_3)$ in Problem 2 is connected. [4 marks]

4. [25 marks]

(a) Define the sum and the product of quaternions. For a quaternion q, define the conjugate quaternion $\overline{q}$ and the modulus $|q|$. Give the formula for $q^{-1} = \frac{1}{q}$ if $q \neq 0$. [5 marks]

(b) Consider the Lie group $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ with the operation of product of quaternions. Identify $Sp(1)$ with the unit sphere S^3 in $\mathbb{R}^4$. [2 marks]

(c) For $q \in Sp(1)$, consider the map

$$\rho(q) : \operatorname{Im} \mathbb{H} \to \operatorname{Im} \mathbb{H}$$

such that $\rho(q)(x) = qx\overline{q}$, $x \in \operatorname{Im} \mathbb{H}$. Here $\operatorname{Im} \mathbb{H}$ is the set of purely imaginary quaternions $bi + cj + dk$. Prove that $\rho(q)$ preserves the modulus $|x|$ and hence defines an orthogonal transformation of $\operatorname{Im} \mathbb{H}$. Prove that it gives a homomorphism of Lie groups $\rho : Sp(1) \to O(3)$. [6 marks]

(d) Prove that the map $\rho : Sp(1) \to O(3)$ has image $SO(3)$ and kernel $\{1, -1\}$. Hence conclude that $SO(3) = Sp(1)/\{1, -1\}$. [7 marks]

(e) By considering paths in $Sp(1)$ and closed paths in $SO(3)$, give arguments that the Lie groups $Sp(1)$ and $SO(3)$ are not isomorphic. [5 marks]

5. [25 marks]

(a) Formulate the functoriality property of Lie algebras of Lie groups under homomorphisms of Lie groups. [3 marks]

(b) Find the Lie algebra $\mathfrak{g}$ of the Lie group

$$G = \left\{ X = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \in Mat_3(\mathbb{R}) \mid \det(X) = 1 \right\} \subset GL(3, \mathbb{R}).$$

[6 marks]

(c) Find the centre

$$c = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$$

of the Lie algebra $\mathfrak{g}$. [7 marks]

(d) What connected Lie subgroup $C_0 \subset G$ corresponds to the Lie subalgebra $c \subset \mathfrak{g}$? [4]

(e) What can you say about the centre of G? [5 marks]

6. [25 marks]

(a) State the functoriality property of the exponential map for a Lie group under homomorphisms of Lie groups. [5 marks]

(b) Let $k = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$,

$$\mathfrak{g} = \{X \in Mat_n(k) \mid \overline{X^t} + X = 0\},$$

and

$$G = \{Y \in Mat_n(k) \mid \overline{Y^t} Y = E\}.$$

Using properties of e^X and $Log(Y)$ for matrices, prove that G is a Lie group and $\mathfrak{g}$ is its Lie algebra. Remark that for $k = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ it gives the Lie groups

$$G = O(n),\ U(n),\ \text{and } Sp(n)$$

and their Lie algebras $\mathfrak{g} = o(n)$, $u(n)$ and $sp(n)$ respectively. [10 marks]

(c) For any square real matrices X, $Y \in gl(n, \mathbb{R}) = Mat_n(\mathbb{R})$ prove the identity

$$e^Y X e^{-Y} = \sum_{n=0}^{\infty} \frac{[Y, [Y, [Y, \ldots, [Y, X]] \ldots](n \text{ times})}{n!}.$$

[10 marks]

7. [25 marks]

(a) Consider the torus $H \subset SO(4) = SO(4, \mathbb{R})$

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix}$$

where $\phi, \psi \in \mathbb{R}$. Prove that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = so(4)$ of H is

$$\mathfrak{h} = \{x_1\sigma_1 + x_2\sigma_2 \mid x_1, x_2 \in \mathbb{R}\},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Hence, or otherwise, prove that the subalgebra $\mathfrak{h}$ is commutative. [6 marks]

(b) Say what it means for a linear function $\alpha : \mathfrak{h} \to \mathbb{R}$ to be a *root* of $\mathfrak{h} \subset \mathfrak{g} = so(4)$, and give the definition of the root subspace $\mathfrak{g}_\alpha \subset \mathbb{C}\mathfrak{g}$. Prove that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. [5 marks]

(c) Prove that the Lie algebra $\mathbb{C}\mathfrak{g} = \mathbb{C}so(4)$ has the Cartan decomposition

$$\mathbb{C}\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathbb{C}\mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$$

where $\pm\alpha_1$, $\alpha_1(x_1\sigma_1 + x_2\sigma_2) = x_1 + x_2$, and $\pm\alpha_2$, $\alpha_2(x_1\sigma_1 + x_2\sigma_2) = x_1 - x_2$, are all roots of $\mathfrak{h} \subset \mathfrak{g}$ with the corresponding root subspaces in $\mathbb{C}\mathfrak{g}$

$$\mathfrak{g}_{\alpha_1} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}; \qquad \mathfrak{g}_{-\alpha_1} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix};$$

$$\mathfrak{g}_{\alpha_2} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ -1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}; \qquad g_{-\alpha_2} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}.$$

[10 marks]

(d) Let $\lambda, \mu \in \{\pm\alpha_1, \pm\alpha_2\}$. For which λ, μ can you say that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = 0$? [4 marks]