Time allowed: 3 hours.

You may attempt as many problems as you like. The best FOUR answers will be taken into account. Each question carries the same weight.

1. [25 marks]

For the real matrix

$$J = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}$$

where E_m is $m \times m$ identity matrix and 0 is $m \times m$ zero matrix consider the set of matrices (the real symplectic group)

$$Sp(m, \mathbb{R}) = \{X \in Mat_{2m}(\mathbb{R}) \mid X^t J X = J\}$$

where X^t is the transpose matrix to X.

(a) Prove that $Sp(m, \mathbb{R}) \subset GL(2m, \mathbb{R})$ is a subgroup. [5 marks]

(b) For the Cayley transform $X = (E - Y)(E + Y)^{-1}$, where $Y \in Mat_{2m}(\mathbb{R})$ and Y is near 0, show that the matrix $X \in Sp(m, \mathbb{R})$, if and only if

$$Y^t J = -JY.$$

[10 marks]

(c) Prove that $Sp(m, \mathbb{R})$ is a Lie group and that $\dim Sp(m, \mathbb{R}) = 2m^2 + m$. [10 marks]

2. [25 marks]

Let V be an n-dimensional vector space over $\mathbb{C}$.

For $0 \leq m_1 \leq m_2 \leq n$, consider the Flag variety

$$Fl(n; m_1, m_2) = \{W_1 \subset W_2 \subset V \mid \dim W_1 = m_1, \dim W_2 = m_2\},$$

where W_1 and W_2 are vector subspaces of V.

(a) By considering the action of the group $GL(V)$ on V, prove that $Fl(n; m_1, m_2)$ is a manifold. Find the dimension of $Fl(n; m_1, m_2)$. [15 marks]

(b) By considering the action of the unitary group $U(n)$, prove that $Fl(n; m_1, m_2)$ is compact. [7 marks]

(c) Use (a) to find the dimension of the space

$$\{\text{line} \subset \text{plane} \subset \text{3-dimensional affine space over } \mathbb{C}\}.$$

[3 marks]

3. [25 marks]

(i) Give definitions of the following Lie groups, and either prove that they are connected or find their connected components:

(a) $O(n)$, $SO(n)$; [12 marks]

(b) $SU(n)$, $U(n)$. [9 marks]

(ii) Prove that the flag variety $Fl(n; m_1, m_2)$ in the Problem 2 is connected. [4 marks]

4. [25 marks]

(a) Define the sum and the product of quaternions. For a quaternion q, define the conjugate quaternion $\overline{q}$ and the modulus $|q|$. Give the formula for $q^{-1} = \frac{1}{q}$ if $q \neq 0$. [5 marks]

(b) Consider the Lie group $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ with the operation of product of quaternions. Identify $Sp(1)$ with the unit sphere S^3 in $\mathbb{R}^4$. [2 marks]

(c) For $(p, q) \in Sp(1) \times Sp(1)$, consider the map

$$\tau(p, q) : \mathbb{H} \to \mathbb{H}$$

such that $\tau(p, q)(x) = pxq^{-1}$, $x \in \mathbb{H}$. Prove that $\tau(p, q)$ preserves the modulus $|x|$ and hence defines an orthogonal transformation of $\mathbb{H}$. Prove that it gives a homomorphism of Lie groups $\tau : Sp(1) \times Sp(1) \to O(4)$. [6 marks]

(d) Prove that the map $\tau : Sp(1) \times Sp(1) \to O(4)$ has image $SO(4)$ and the kernel which is the group of order two $\{(1, 1)\,(-1, -1)\}$. Hence conclude that $SO(4) = (Sp(1) \times Sp(1))\,/\{(1, 1), (-1, -1)\}$. [7 marks]

(e) By considering paths in $Sp(1) \times Sp(1)$ and closed paths in $SO(4)$, give arguments that the Lie groups $Sp(1) \times Sp(1)$ and $SO(4)$ are locally isomorphic, but not isomorphic. [5 marks]

5. [25 marks]

(a) Formulate the functoriality property of Lie algebras of Lie groups under homomorphisms of Lie groups. [4 marks]

(b) Find the Lie algebra $\mathfrak{g}$ of the Lie group

$$T = \{X = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \mid \det(X) \neq 0\} \subset GL(2, \mathbb{R}).$$

of upper triangular 2×2 real matrices. [8 marks]

(c) Find the Lie algebra $s\mathfrak{g} \subset \mathfrak{g}$ of its special subgroup

$$ST = \{X \in T \mid \det(X) = 1\}. \quad [5 \text{ marks}]$$

(d) Find the centre

$$c = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$$

of the Lie algebra $\mathfrak{g}$. [4 marks]

(e) What connected Lie subgroup in T corresponds to the Lie subalgebra? What can you say about this Lie subgroup? [4 marks]

6. [25 marks]

(a) State the functoriality property of the exponential map for a Lie group under homomorphisms of Lie groups. [5 marks]

(b) Let $k = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$,

$$\mathfrak{g} = \{X \in Mat_n(k) \mid \overline{X^t} + X = 0\},$$

and

$$G = \{Y \in Mat_n(k) \mid \overline{Y^t}Y = E\}.$$

Using properties of e^X and $Log(Y)$ for matrices, prove that G is a Lie group and $\mathfrak{g}$ is its Lie algebra. Remark that for $k = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ it gives the Lie groups

$$G = O(n),\ U(n),\ \text{and } Sp(n)$$

and their Lie algebras $\mathfrak{g} = o(n)$, $u(n)$ and $sp(n)$ respectively. [10 marks]

(c) For a square real matrix A, prove the identity

$$\det(e^A) = e^{\operatorname{tr}(A)}.$$

[10 marks]

7. [25 marks]

(a) Prove that the first two (up to order 2) terms of the Campbell–Baker–Hausdorff series:

$$Log(Exp(X)Exp(Y)) = (X+Y) + \frac{1}{2}[X,Y] + \left(\frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]]\right) + \\ + \frac{1}{24}[Y,[X,[Y,X]]] + \cdots$$

are as shown where the matrices X, $Y \in gl(n,\mathbb{R}) = Mat_n(\mathbb{R})$ are small (i.e. they are near the zero matrix). [11 marks]

(b) Formulate the general statement about the Campbell–Baker–Hausdorff series. [3 marks]

(c) Formulate the general statement about existence of a local linear Lie subgroup $Exp(U)$ of $GL(n,\mathbb{R})$ for a small neighbourhood U of 0 in a linear subspace $\mathfrak{h} \subset gl(n,\mathbb{R})$. Sketch the proof of the statement. [6 marks]

(d) Formulate the general statement about local isomorphism of two linear Lie subgroups $G_1 \subset GL(n,\mathbb{R})$ and $G_2 \subset GL(m,\mathbb{R})$ with isomorphic Lie algebras (here n and m can be different). Sketch the proof of the statement. [5 marks]

8. [25 marks]

(a) Consider the torus $H \subset SO(4) = SO(4, \mathbb{R})$

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix}$$

where $\phi, \psi \in \mathbb{R}$. Prove that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = so(4)$ of H is

$$\mathfrak{h} = \{x_1\sigma_1 + x_2\sigma_2 \mid x_1,\, x_2 \in \mathbb{R}\},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Deduce, or prove otherwise, that the subalgebra $\mathfrak{h}$ is commutative. [6 marks]

(b) Say what it means for a linear function $\alpha : \mathfrak{h} \to \mathbb{R}$ to be a *root* of $\mathfrak{h} \subset \mathfrak{g} = so(4)$, and give the definition of the root subspace $\mathfrak{g}_\alpha \subset \mathbb{C}\mathfrak{g}$. Prove that $[\mathfrak{g}_\alpha,\, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. [5 marks]

(c) Prove that the Lie algebra $\mathbb{C}\mathfrak{g} = \mathbb{C}so(4)$ has the Cartan decomposition

$$\mathbb{C}\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathbb{C}\mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$$

where $\pm\alpha_1$, $\alpha_1(x_1\sigma_1 + x_2\sigma_2) = x_1 + x_2$, and $\pm\alpha_2$, $\alpha_2(x_1\sigma_1 + x_2\sigma_2) = x_1 - x_2$, are all roots of $\mathfrak{h} \subset \mathfrak{g}$ with the corresponding root subspaces in $\mathbb{C}\mathfrak{g}$

$$\mathfrak{g}_{\alpha_1} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}; \qquad \mathfrak{g}_{-\alpha_1} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix};$$

$$\mathfrak{g}_{\alpha_2} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ -1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}; \qquad g_{-\alpha_2} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}.$$

[10 marks]

(d) Let $\lambda, \mu \in \{\pm\alpha_1, \pm\alpha_2\}$. For which λ, μ can you say that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = 0$? [4 marks]