2MP75 (MATH445) January 1998 Examination

Time allowed: Two Hours and a Half

The best FIVE questions will be credited

1. Define a *diffeomorphism* between subsets of Euclidean spaces.

Show that the graph of a smooth map $h : \mathbf{R}^k \to \mathbf{R}^n$ is diffeomorphic to $\mathbf{R}^k$.

Give the definition of a *manifold*.

Show that the sphere $S^3 = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ in the real coordinate 4-dimensional space $\mathbf{R}^4$ is a 3-dimensional manifold.

Define the *tangent space* to a manifold at a point.

Exhibit a basis for the tangent space to the sphere S^3 at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

[20 marks]

2. Formulate the *Inverse Function Theorem*.

What is a *submersion at a point*?

Formulate the *Local Submersion Theorem*.

Prove that the preimage of a regular value of a mapping is a submanifold in the source. What is its dimension?

Show that 0 is the only critical value of the determinant function on the space $M(2)$ of all 2×2 matrices with real entries. Thus prove that the group $SL(2)$ of all 2×2 matrices with determinant 1 is a submanifold of $M(2)$ and hence a Lie group. What is the dimension of $SL(2)$? What is the tangent space to $SL(2)$ at the identity matrix? [20 marks]

3. Formulate *Sard's Theorem*. Formulate the *Morse Lemma*.

Show that, for a smooth function f on $\mathbf{R}^k$, the function

$$f + a_1 x_1 + \ldots + a_k x_k$$

is Morse for almost all k-tuples of constants $a_1, \ldots, a_k$.

Find all the critical points of the functions f below. Which of the points are Morse? Which of the functions are Morse? In each of the cases when f is not Morse, give an example of a linear function ℓ on the Euclidean space containing the domain of f such that $f + \ell$ is Morse on the domain.

a) $f = \frac{1}{2}x^2 + \frac{1}{4}y^4 - \frac{1}{3}y^3$ on $\mathbf{R}^2$;

b) $f = x$ on the cylinder $S^1 \times \mathbf{R}^1 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = 1\}$.

[20 marks]

4. Define a *manifold with boundary*.

What does it mean that a mapping f of a manifold X with boundary into a manifold Y is *transversal* to a submanifold $Z \subset Y$? Here Y and Z are assumed to have no boundaries. What can one say about $f^{-1}(Z)$ in these circumstances?

Determine for which values of a the intersection of the ball $x^2 + y^2 + z^2 \leq 1$ with the hyperboloid $x^2 + y^2 - z^2 = a$ is a 2-dimensional manifold with boundary.

[20 marks]

5. Define the *mod 2 intersection number* $I_2(f, Z)$ of a map $f : X \to Y$ with a submanifold Z.

Outline the proof of the fact that the 2-sphere is not diffeomorphic to any surface of positive genus. You do not need to prove the details.

Define the *mod 2 winding number* of a mapping into an Euclidean space around a point.

Formulate the *Borsuk-Ulam Theorem*.

Prove that the functions

$$x \cos y + y \cos z + z \cos x \qquad \text{and}$$

$$(x^2 + 3y^2) \tan z + (y^2 + 4z^2) \tan x + (z^2 + 5x^2) \tan y$$

have at least 2 common zeros on the 2-sphere $x^2 + y^2 + z^2 = 1$.

[20 marks]

6. Define an *oriented* manifold (with boundary).

Let $B^3 \subset \mathbf{R}^3$ be the unit ball with the standard orientation of $\mathbf{R}^3$. Consider $S^2 = \{x^2+y^2+z^2 = 1\} = \partial(B^3)$ with the boundary orientation. Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be the mapping $(x, y) \mapsto (x, y, 0)$. Assuming that $\mathbf{R}^2$ is also equipped with its standard orientation, find the induced preimage orientation on $f^{-1}(S^2)$.

Define the *oriented* intersection number of a mapping with a submanifold.

Define the intersection number $I(X, Z)$ of two compact submanifolds.

Define the *degree* of a smooth map.

Determine the degree of the map

$$z \mapsto \frac{\sin z}{|\sin z|}$$

from the circle of radius 10 centred at $0 \in \mathbf{C}$ to the unit circle in $\mathbf{C}$. Both circles are oriented counter-clockwise. [20 marks]

7. Define the *global Lefschetz number* $L(f)$ of a map f.

Formulate the *Smooth Lefschetz Fixed-Point Theorem.*

How can one calculate the local Lefschetz number $L_x(f)$ at a Lefschetz fixed point x in terms of the matrix of df_x?

Find the Euler characteristic of S^k. Thus prove that the antipodal map

$$x \mapsto -x$$

of an even-dimensional sphere into itself is not homotopic to the identity.

What is the *local Lefschetz number* of an arbitrary isolated fixed point of a map in terms of degrees of mappings? Find the local Lefschetz number at the origin of the map $z \mapsto z + 2\bar{z}^m$ of $\mathbf{C} = \mathbf{R}^2$ into itself, $m \geq 0$. [20 marks]

8. Define a *vector field* on a manifold.

What is the *index* of a fixed point of a vector field? Is the index defined for any fixed point?

What is the index of the only zero of the vector field $v(z) = iz^m$, $m > 0$, on $\mathbf{C} = \mathbf{R}^2$? Here $i = \sqrt{-1}$.

Formulate the *Poincaré-Hopf Index Theorem.*

What does it give for a genus $g \geq 1$ surface?

Consider a vector field on a genus $g \geq 1$ surface with all its fixed points non-degenerate. What is the minimal possible number of fixed points this vector field may have? What are the types of the fixed points in the minimal case? Explain (just qualitatively) how to construct a 'minimal non-degenerate' vector field on a

a) torus,

b) surface of arbitrary genus $g > 1$. [20 marks]