MATH445 January 2000 Examination

Time allowed: Three Hours

The best FIVE questions will be credited

1. (a) Define a *diffeomorphism* between subsets of Euclidean spaces.

Consider the hyperboloid $H \subset \mathbf{R}^3$ given by the equation

$$x^2 + y^2 - z^2 = -1\,.$$

Consider the projection $P : \mathbf{R}^3 \to \mathbf{R}^2$, $P(x, y, z) = (x, y)$. Show that P defines a diffeomorphism between the part of H contained in the half-space $z < 0$ and $\mathbf{R}^2$.

(b) Give the definition of a *manifold*.

Show that the hyperboloid H from part (a) is a 2-dimensional manifold.

(c) Define the *tangent space* to a manifold at a point.

Exhibit a basis for the tangent space to the above hyperboloid H at the point $(0, 0, -1)$.

[20 marks]

2. What is a *submersion at a point*?

Formulate the *Local Submersion Theorem*.

Prove that the preimage of a regular value of a mapping is a submanifold in the source. What is its dimension?

Let $M(3)$ be the space of all 3×3-matrices with real entries, and $S(3)$ its subspace of all symmetric matrices. Consider the mapping $f : M(3) \to S(3)$, $f(A) = AA^T$. Show that the unit matrix I_3 is a regular value of f. Thus prove that the orthogonal group $O(3)$ is a submanifold of $M(3)$. What is the dimension of $O(3)$?

[20 marks]

3. What is a *Morse critical point* of a function on $\mathbf{R}^n$?

Formulate the *Morse Lemma.*

Find all the critical points of the function

$$f(x,y) = (x-y)^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 \ .$$

Specify which of the critical points you found are Morse and which of these Morse points are local minima or maxima.

Formulate *Sard's Theorem.*

Change the above function, by adding an appropriate linear function, to a function with Morse critical points only.

[20 marks]

4. Define a *manifold with boundary*.

What does it mean to say that a mapping f of a manifold X with boundary into a manifold Y is *transversal* to a submanifold $Z \subset Y$? Here Y and Z are assumed to have no boundaries. What can one say about $f^{-1}(Z)$ and its codimension in X in this case?

Determine for which values of a the intersection of the unit ball $x^2+y^2+z^2 \leq 1$ with the hyperboloid $x^2+y^2-2z^2 = a$ is a non-empty 2-dimensional manifold with boundary. For which values of a is this manifold connected?

[20 marks]

5. (a) Define the *mod 2 intersection number* $I_2(f, Z)$ of a map $f : X \to Y$ with a submanifold Z.

Let $i : S^1 \to T^2$ be the inclusion map of a meridian into the torus. Show that it is not homotopic to a constant map of S^1 to T^2.

(b) Define the *mod 2 winding number* of a mapping into an Euclidean space around a point.

Formulate the *Borsuk-Ulam Theorem*.

Prove that the functions

$$x \sec y + y \sec z + z \sec x \qquad \text{and}$$
$$(x^2+1)\sin z + (y^2+2)\sin x + (z^2+3)\sin y$$

have at least two common zeros on the 2-sphere $x^2 + y^2 + z^2 = 1$.

[20 marks]

6. (a) Define what it means for a manifold (with boundary) to be *oriented*.

Define the *oriented intersection number* $I(X, Z)$ of two compact submanifolds. Assuming X transversal to Z show (directly from the definition) that

$$I(Z, X) = (-1)^{(\dim X)\cdot(\dim Z)} I(X, Z) \ .$$

Hence find the self-intersection number of a $(2k+1)$-dimensional submanifold in a $(4k+2)$-dimensional manifold.

(b) Define the *degree* of a smooth map.

Determine the degree of the map

$$z \mapsto \frac{\cos 2z}{|\cos 2z|}$$

from the circle of radius 3 centred at $0 \in \mathbf{C}$ to the unit circle in $\mathbf{C}$. Both circles are oriented counter-clockwise.

[20 marks]

7. Define the *global Lefschetz number* $L(f)$ of a map f.

Formulate the *Smooth Lefschetz Fixed-Point Theorem.*

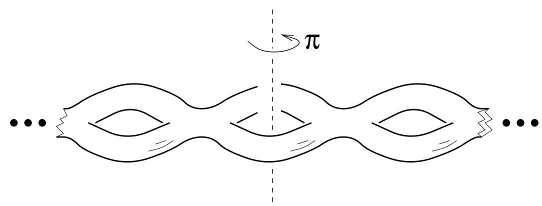
What is the global Lefschetz number of a mapping $f : X \to X$ homotopic to the identity?

How can one calculate the *local* Lefschetz number $L_x(f)$ at a *Lefschetz* fixed point x in terms of the matrix of df_x? [The proof of the statement is not required.]

Calculate the Euler characteristic of S^2.

Recall, without any calculations, the numerical value of the Euler characteristic of a surface of genus g.

Consider the surface of odd genus, and its rotation through π around its symmetry axis passing through the central hole (see the figure). Show that the mapping thus obtained of the surface into itself is not homotopic to the identity if $g > 1$.



[20 marks]

8. Define a *vector field* on a manifold.

What is the *index* of an isolated fixed point of a vector field?

Formulate the *Poincaré-Hopf Index Theorem.*

Calculate the Euler characteristic of the Klein bottle constructing an appropriate vector field on it.

Consider a vector field on a S^k with all its fixed points *non-degenerate.* What is the minimal possible number of fixed points this vector field may have? [You may assume the Euler characteristic of S^k known.] Give examples of such minimal vector fields on odd- and even-dimensional spheres. Make sure your examples cover all positive dimensions.

[20 marks]