1. (a) Let γ be a regular plane curve with parameter t and arc-length $s(t)$. Explain briefly the meaning of the standard formulae

$$\gamma' = Ts', \quad T' = \kappa N s' \quad \text{(where } ' \text{ means } d/dt)$$

and deduce that $N' = -\kappa T s'$. Writing $\gamma(t) = (X(t), Y(t))$ derive the formula $\kappa = (X'Y'' - X''Y')/(X'^2 + Y'^2)^{3/2}$ for the curvature $\kappa(t)$.

Now let $\gamma(t) = (t^2, t^5)$. Show that γ is regular for $t \neq 0$ and that, for $t \neq 0$,

$$\kappa(t) = \frac{30|t|}{(4 + 25t^6)^{3/2}}.$$

Indicate by a sketch of the curve why $\kappa(t)$ is always > 0 for $t \neq 0$. [15 marks]

(b) Explain briefly the connexion between height functions on a plane curve and contact with lines.

Write down the height function corresponding to the unit vector (a, b) for the curve $\gamma(t) = (t^2, t^4 + t^5)$. Show that γ is regular for $t \neq 0$ and find the value or values of $t \neq 0$ for which γ has an ordinary inflexion. [10 marks]

2. Let α be a unit speed space curve whose curvature is never zero. Write down the standard formulae for T', N', B' where T is the unit tangent, N the unit principal normal and B the unit binormal of α, and $'$ denotes differentiation with respect to arclength s.

Explain briefly the connexion between distance-squared functions on a space curve and contact with spheres. Let α be a unit speed space curve whose curvature is never zero and let $\alpha(s)$ be a point at which the torsion $\tau(s)$ is nonzero. Show that there is a unique sphere having at least 4-point contact with α at $\alpha(s)$, and its centre is the point

$$u = \alpha(s) + \frac{1}{\kappa(s)} N(s) - \frac{\kappa'(s)}{\kappa(s)^2 \tau(s)} B(s).$$

What is the corresponding result at points $\alpha(s)$ where $\tau(s) = 0$? [25 marks]

3. (a) State the meaning of the phrase '$f : \mathbf{R}, t_0 \to \mathbf{R}$ has an A_k singularity at t_0'.

Find values of the constants a and b such that

$$f(t) = \tfrac{1}{5}t^5 + \tfrac{1}{4}at^4 + \tfrac{1}{3}bt^3$$

has A_1 singularities at $t = 1$ and $t = 2$.

What singularity A_k does f then have at $t = 0$? Find an explicit local diffeomorphism $h : \mathbf{R}, 0 \to \mathbf{R}, 0$ such that $f(t) = \pm(h(t))^{k+1}$. State briefly why your h is a local diffeomorphism. [12 marks]

(b) Let $\phi : \mathbf{R}^2 \to \mathbf{R^2}$ be given by $\phi(x, y) = (w, z) = (x, x^2y + y^2)$. Write down the Jacobian matrix J of ϕ and sketch in the (x, y) plane the critical set Σ of ϕ, where $\det(J) = 0$. Sketch also the set of points $\phi(x, y)$ for $(x, y) \in \Sigma$.

Find all the points (x, y) for which $\phi(x, y) = (1, 6)$. What does the inverse function theorem say about local inverses of ϕ for (w, z) close to $(1, 6)$? For each such local inverse, find $\partial y/\partial w$ and $\partial y/\partial z$ at $(w, z) = (1, 6)$. [13 marks]

4. (a) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = x^2 + 2xy^2 - y^2 + y^3 + y^4.$$

Show that the only critical points of f are $(0, 0)$ and $(-\frac{4}{9}, \frac{2}{3})$. Deduce that $f^{-1}(0) - \{(0, 0)\}$ is, in a neighbourhood of any of its points, a parametrized 1-manifold, stating carefully any general result you use. Parametrizing by x or y as appropriate, find the curvature at $(-1, 1)$. [13 marks]

(b) Let $\alpha : I \to \mathbf{R}^2$ be a unit speed plane curve, with arclength parameter t. Write $\alpha(t) = (X(t), Y(t))$. Define a map

$$F : I \times \mathbf{R^2} \to \mathbf{R} \text{ by } F(t, x, y) = ((x, y) - \alpha(t)).T(t),$$

where T as usual is the unit tangent $(X'(t), Y'(t))$ to α. Define

$$G : I \times \mathbf{R}^2 \to \mathbf{R}^2 \text{ by } G(t, x, y) = \left(F(t, x, y), \frac{\partial F}{\partial t}(t, x, y)\right).$$

Use the implicit function theorem to show that $G^{-1}(0, 0)$ is a parametrized 1-manifold in a neighbourhood of any of its points, and that t can always be used as a local parameter. [12 marks]

5. Throughout this question, $\gamma : I \to \mathbf{R}^2$ is a regular plane curve which does not pass through the origin (0,0). The curvature of γ will be denoted by κ. The 'orthotomic' of γ relative to the origin is the curve $\delta(t) = 2(\gamma(t).N(t))N(t)$, where N is as usual the unit normal to γ.

(i) In this part, you may assume

- γ is unit speed;
- $\gamma(t).N(t)$ is never zero. (Geometrically this means that no tangent to γ passes through the origin; you need not verify this.)

Show that the equation of the circle through (0,0), centre $\gamma(t)$, is

$$(\mathbf{x} - 2\gamma(t)).\mathbf{x} = 0.$$

Show that the envelope of these circles consists of the origin together with the orthotomic of γ relative to the origin.

Show that the envelope is regular at points $\mathbf{x} \neq (0,0)$ where $\mathbf{x} = \delta(t)$ and $\kappa(t) \neq 0$.

Use the versal unfolding method to show that there is an ordinary cusp on the envelope at $\mathbf{x} = \delta(t)$ provided $\kappa(t) = 0, \kappa'(t) \neq 0$. [16 marks]

(ii) Let $\gamma(t) = (t-1, f(t))$, where f is smooth, $f(0) = f'(0) = 0$ and $f''(0) \neq 0$. Find an explicit parametrization for the orthotomic $\delta(t) = (X(t), Y(t))$, say, in terms of f and f'.

Show that $\delta(0) = (0,0)$ and that the tangent to γ at $\gamma(0)$ passes through the origin.

Show further that $Y'(0) \neq 0$, and deduce that δ is regular at $t = 0$.
[9 marks]

6. (a) In each of the following cases, the formula F gives an unfolding of the function $f(t) = F(t, \mathbf{0})$ at $t = 0$. Determine the A_k type of the function f at $t = 0$ and whether the unfolding is versal.

$$\begin{aligned} F(t, x, y) &= t^3 + xyt^2 + (2x + 3y)t + y, \\ F(t, x, y, z) &= t^4 + xt^3 + xyt^2 + (2x + z)t + y. \end{aligned}$$

[8 marks]

(b) Let α and β be unit speed plane curves (arclength parameters s and t respectively). We write T_α, N_α for the unit tangent and normal to α, and similarly for β. Suppose

- $\alpha(s)$ never equals $\beta(t)$ (i.e. the curves are disjoint);
- $T_\alpha(s)$ never equals $T_\beta(t)$.

Show that $T_\alpha.N_\beta + T_\beta.N_\alpha = 0$. [Hint: it might help to remember that if $T = (a, b)$ then $N = (-b, a)$.] Deduce that $T_\alpha - T_\beta$ is perpendicular to $N_\alpha - N_\beta$.

Consider the map $F(s, t) = (\alpha(s) - \beta(t)).(T_\alpha(s) - T_\beta(t))$. Show that $F(s, t) = 0$ if and only if there exists a real number $r \neq 0$ such that $\alpha(s) + rN_\alpha(s) = \beta(t) + rN_\beta(t)$.

Show also that 0 is a regular value of F unless both the radii of curvature of α at $\alpha(s)$ and β at $\beta(t)$ equal r. [17 marks]

7. Let $\alpha : I \to \mathbf{R}^3$ be a unit speed space curve (arclength parameter t, say) with $\kappa(t)$ never zero *and also the torsion* $\tau(t)$ *never zero.*

Define $F : I \times \mathbf{R}^3 \to \mathbf{R}$ by

$$F(t, \mathbf{x}) = (\mathbf{x} - \alpha(t)).B(t),$$

where B is as usual the binormal vector. For a fixed t, what is the set of points $F(t, \mathbf{x}) = 0$?

Show that the envelope of the family F is precisely the points $\mathbf{x} = \alpha(t) + \lambda T(t)$ for $\lambda \in \mathbf{R}$, where T is as usual the unit tangent vector to α.

Find the points of regression on the envelope and show (using $\tau(t) \neq 0$) that these are always of type A_2 and that they are always versally unfolded by the family F.

Now let $\alpha(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$. Show that α satisfies the conditions $\kappa(t) \neq 0$ and $\tau(t) \neq 0$ and use the above result to determine the local structure of the envelope of F at all points of regression. [25 marks]