1. (a) Define the *distance-squared function* f from the point (a, b) to the plane curve $\gamma : I \to \mathbb{R}^2$. Explain briefly the connexion between the number of derivatives of f which vanish at t and the contact of a circle, centre (a, b), with γ at $\gamma(t)$. [4 marks]

(b) Define the *height function* h in the direction $(a, b) \neq (0, 0)$ for a plane curve γ. Explain briefly the connexion between the height function and the inflexions or higher inflexions of γ. [4 marks]

(c) Now let λ be a nonzero real number and let $\gamma(t) = (t^2, \lambda t + t^4)$. Show that γ is a regular curve. Write down the corresponding distance-squared function and show that there is a unique circle having exactly 4-point contact with γ at the origin; in particular, find the centre of this circle in terms of λ.

For the same curve γ, use the height function to show that γ has exactly one inflexion, and that it is not a higher inflexion. [17 marks]

2. (a) Let α be a unit speed space curve. Define the unit tangent T, the curvature κ and, assuming $\kappa \neq 0$, the unit principal normal N, the unit binormal B and the torsion τ, stating expressions for T', N' and B' in terms of T, N, B, κ, τ. [6 marks]

(b) Let $\alpha(s) = (\frac{4}{5}\cos s, \frac{3}{5}s, \frac{4}{5}\sin s)$. Show that α is unit speed and find, in terms of s, the unit tangent, principal normal, binormal, curvature and torsion. [8 marks]

(c) Let the space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ be defined by $\gamma(t) = (t + 4t^4, t^3, t + 4t^5 - t^6)$. Writing x, y, z for the coordinates in $\mathbb{R}^3$, show that γ meets the plane $x - z = 0$ in the points with parameters $t = 0$ and $t = 2$. Write down the height function on γ in the direction $(1, 0, -1)$ and use it to determine, for $t = 0$ and $t = 2$, the contact between γ and the plane $x - z = 0$. State the criterion you are using to determine contact. [11 marks]

3. (a) Write down the meaning of the phrase 'the function f (with variable t) has an A_k singularity at $t = t_0$'.

Let $f(t) = t^4 + at^3 + bt^2$. Find the values of a and b such that f has an A_1 singularity at $t = 0, t = 1$ and $t = -2$. (You must verify that your f does have exactly these singularities.)

Find a local diffeomorphism $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ such that $f(t) = \pm(h(t))^k$, for an appropriate value of k, and all t close to 0. State briefly why your h is a local diffeomorphism. [13 marks]

(b) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\phi(x, y) = (x - y^2, x^2 + y^3) = (w, z)$ say. Find the critical set Σ of f, that is the set of points (x, y) for which the Jacobian matrix of f has zero determinant. Sketch Σ on a diagram.

Find all points (x, y) for which $\phi(x, y) = (0, 0)$, and mark them on your diagram. Do any of these points lie on Σ?

What does the Inverse Function Theorem say about local inverses of ϕ, defined near $(w, z) = (0, 0)$? For any such local inverse, determine the values of $\partial x/\partial w$ and $\partial y/\partial w$ at $(w, z) = (0, 0)$. [12 marks]

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^3 + x^2y^2 + y^4$.

(a) Show that the only critical points of f are $(0, 0)$ and $(0, -\frac{3}{4})$. Let $C = f^{-1}(0) - \{(0, 0)\}$. Show that C is a parametrized 1-manifold in a neighbourhood of any of its points, stating clearly any general result which you use. Parametrizing C by x or y as appropriate, find the curvature of C at $(0, -1)$. State without proof any formula which you use for curvature. [15 marks]

(b) Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $g(x, y, z) = (z - f(x, y),\ x - y^2)$, for the same f as above. Show that $g^{-1}(0, 0)$ is a parametrized 1-manifold in a neighbourhood of any of its points. Verify that $(1, 1, 4)$ is a point of $g^{-1}(0, 0)$ and find a nonzero tangent vector at this point. [10 marks]

5. Throughout this question, α is a unit speed plane curve with unit tangent T, unit normal N, and curvature κ *never* 0.

Show that the equation of the tangent to α at $\alpha(t)$ is $(\mathbf{x} - \alpha(t)) \cdot N(t) = 0$ and deduce that this tangent passes through the origin $\mathbf{x} = \mathbf{0}$ if and only if $\alpha(t) \cdot N(t) = 0$.

Assume from now on that $\alpha(t) \cdot N(t)$ is never 0.

Now let $F(t, \mathbf{x}) = \alpha(t) \cdot (2\mathbf{x} - \alpha(t))$. Show that $F(t, x) = 0$ if and only if the distance of $\mathbf{x}$ from $\alpha(t)$ equals the distance of $\mathbf{x}$ from the origin $\mathbf{0}$. (It follows that $F(t, \mathbf{x}) = 0$ is the equation of the perpendicular bisector of the line from the origin to $\alpha(t)$ but you need not show this.)

Show that $\frac{\partial F}{\partial t} = 2T(t) \cdot (\mathbf{x} - \alpha(t))$ and deduce that the envelope of F consists of points $\mathbf{x} = \alpha(t) + \lambda N(t)$ where

$$\lambda = -\frac{\alpha(t) \cdot \alpha(t)}{2\alpha(t) \cdot N(t)}.$$

Show that $\mathbf{x}$ is a point of regression on the envelope if and only if $\kappa\lambda = 1$ for the above value of λ.

Show that the condition for $F_{\mathbf{x}}$ to have exactly an A_2 singularity at the parameter value t is that in addition $\kappa'(t) \neq 0$.

Show finally that the versal unfolding condition is always satisfied for A_2 singularities and deduce the local structure of the envelope at such points $\mathbf{x}$.

[25 marks]

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6. (a) Let $\alpha : I \to \mathbb{R}^2$ be a unit speed plane curve, where $\alpha(t) = (X(t), Y(t))$. Explain why the unit normal $N(t)$ is $(-Y'(t), X'(t))$.

Let two maps $I \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ be defined by

$$\gamma(t,u) = (\alpha(t) + uN(t), t), \qquad \delta(t,u) = (\alpha(t) + uN(t), u).$$

Show that γ is an immersion at all points (t,u), and that δ fails to be an immersion precisely at points where $\kappa(t) \neq 0$ and $u = \dfrac{1}{\kappa(t)}$ (as usual, κ is the curvature of α). [15 marks]

(b) In each of the following cases, the formula F gives an unfolding of the function $f(t) = F(t, \mathbf{0})$ at $t = 0$. Determine the A_k type of the function f at $t = 0$ and whether the unfolding is versal. State the versality criterion which you are using.

$$\begin{array}{lrcl} \text{(i)} & F(t,x,y) & = & t^3 + (2x+3y)t + (x^2+y), \\ \text{(ii)} & F(t,x,y,z) & = & -t^5 + (\cos x)t^4 + (\sin y)t^2 + xt + z. \end{array}$$

[10 marks]

7. Let γ be a unit speed space curve with curvature never zero. The *normal plane* to γ at the parameter value t is the plane through $\gamma(t)$ orthogonal to $T(t)$, i.e. with equation $F(\mathbf{x}, t) = 0$, where $F(\mathbf{x}, t) = (\mathbf{x} - \gamma(t)) \cdot T(t)$, where $\mathbf{x} \in \mathbb{R}^3$.

Show that the envelope of these normal planes contains one line in each normal plane, having the form

$$\mathbf{x} = \gamma(t) + \frac{1}{\kappa(t)} N(t) + \mu B(t),$$

where μ is an arbitrary real number.

Show that the points of regression on the envelope are given by either (a) $\tau = \kappa' = 0$, μ arbitrary, or (b) $\tau \neq 0$, $\mu = -\dfrac{\kappa'}{\kappa^2\tau}$, where κ, κ', τ are evaluated at t.

Find the 1-jet and 2-jet matrices with constants for the unfolding F. Show that the 1-jet matrix always has rank 2 and the 2-jet matrix has rank 3 if and only if $\tau \neq 0$.

State what can be deduced about the structure of the envelope of normal planes at $\mathbf{x}$, when (i) F has type A_2, (ii) F has type A_3. (You need not calculate the conditions for these A_k types to occur.) [25 marks]