1.

The Navier-Stokes equation of motion can be written as follows:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}.$$

(i) [7 marks]

Non-dimensionalise this equation based on characteristic length scale, L and velocity scale, U, and scale pressure based on the viscous terms. What is the Stokes flow approximation?

(ii) [5 marks]

Assuming Stokes flow is a valid approximation, explain why the drag force, F, on a body of length L sinking at speed U in water of viscosity μ can be written:

$$F = C_f \mu L U,$$

where C_f is a dimensionless number which only depends on the shape of the body.

(iii) [8 marks]

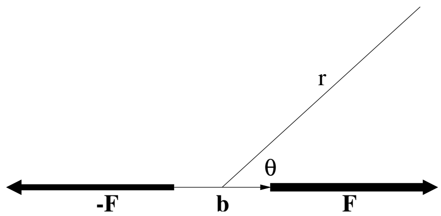
To compute the sinking speed of a diatom with a complex geometry, a self-similar model is constructed with length 1cm. This model diatom has the same density as a real diatom, and is measured to sink at 5cm/s in glycerol solution ($\rho = 1\text{g/cm}^3, \mu = 1000\text{g/cms}$). If the real diatom has length 20×10^{-4}cm, what speed does it sink at in seawater ($\rho = 1\text{g/cm}^3, \mu = 10^{-2}\text{g/cms}$)?

2. Assume Stokes flow throughout.

The stresslet flow field is given by the equation:

$$\mathbf{u}_{stress} = \frac{1}{8\pi\mu}\left(-\frac{\mathbf{F}.\mathbf{b}}{r^3} + 3\frac{(\mathbf{F}.\mathbf{r})(\mathbf{b}.\mathbf{r})}{r^5}\right)\mathbf{r}$$

This flow field can be generated by 2 forces of equal and opposite strength ($\mathbf{F}$ and $-\mathbf{F}$) separated by the vector $\mathbf{b}$ as shown in the following figure:



(i) [4 marks]

In spherical polar coordinates (r, θ, ϕ) as shown in the figure, show that the stresslet flow field reduces to

$$\mathbf{u}_{stress} = \frac{|\mathbf{F}||\mathbf{b}|}{8\pi\mu r^2}(3\cos^2\theta - 1)\mathbf{e}_r, \quad \text{where} \quad \mathbf{r} = r\mathbf{e}_r. \tag{1}$$

(ii) [4 marks]

Verify that the flow field described by eq. (1) is incompressible. You may use the result that

$$\nabla.\mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

(iii) [12 marks]

Consider a neutrally buoyant self propelled sphere. With the assistance of clear diagrams, explain how a stresslet could be used to model this organism. Using the stresslet model, and taking a frame of reference moving with the organism, thus compute the magnitude of the velocity a distance $10a$ directly ahead of the organism. You may use the result that the drag on a sphere has magnitude $6\pi\mu aU$.

3. Consider a micro-organism which swims at steady velocity $-U\mathbf{i}$ and is propelled by an inextensible flagellum of length L undergoing planar waving motion with wave velocity $V\mathbf{i}$. At $t = 0$ the position of centreline of the flagellum is given by the parametric equation:

$$\mathbf{R}(s) = (X(s), Y(s)), \quad 0 < s < L,$$

where s is arc length measured from one end. The wave is periodic and satisfies

$$X(0) = 0, \quad X(L) = \alpha L.$$

Resistive force theory states that the force acting on an element of flagellum of length ds moving with velocity $-\mathbf{w}$ is given by

$$F_t = K_T \mathbf{w}.\mathbf{t} ds, \quad F_n = K_N \mathbf{w}.\mathbf{n} ds.$$

where F_t and F_n are the components of force tangential and normal to the flagellum respectively, and $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and normal vectors respectively.

(i) [7 marks]

Show that F, the force in the $\mathbf{i}$ direction on an element of flagellum of length ds moving with velocity $-\mathbf{w}$, can be written as

$$F \;=\; (K_T - K_N)(\mathbf{w}.\mathbf{t})(\mathbf{t}.\mathbf{i})ds + K_N \mathbf{w}.\mathbf{i} ds.$$

You can use the identity $\mathbf{w}.\mathbf{i} = (\mathbf{w}.\mathbf{t})(\mathbf{t}.\mathbf{i}) + (\mathbf{w}.\mathbf{n})(\mathbf{n}.\mathbf{i})$.

(ii) [13 marks]

The velocity of a material point on the flagellum relative to the fluid far away is $-\mathbf{w}$, where

$$\mathbf{w} = (U - V)\mathbf{i} + \frac{V}{\alpha}\mathbf{t}.$$

Show that the total thrust in the $\mathbf{i}$ direction generated by the flagellum is given by

$$T = (V - U)[(K_T - K_N)\beta L + K_N L] - K_T V L, \quad \beta L = \int_0^L X'^2 ds.$$

4. A bottom heavy sphere is at position $\mathbf{x}(t) = (x(t), y(t), 0)$ is placed in Poiseuille flow between two parallel plates, $\mathbf{u} = -\frac{W_0}{h^2}(h^2 - x^2)\hat{\mathbf{y}}$. The micro-organism swims at speed v and the swimming direction $\mathbf{p} = (\sin\theta, \cos\theta, 0)$ satisfies the following vector equation:

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}[\hat{\mathbf{y}} - (\hat{\mathbf{y}}.\mathbf{p})\mathbf{p}] + \frac{1}{2}\boldsymbol{\omega} \wedge \mathbf{p},$$

where $\boldsymbol{\omega}$ is the vorticity of the flow.

(i) [11 marks]

Derive a set of differential equations satisfied by $\theta(t), x(t)$ and $y(t)$.

(ii) [3 marks]

What conditions must be satisfied for a there to be a steady-state solution for swimming direction? (You may assume the cell's swimming speed is sufficiently small for variation in the ambient flow as the cell swims to be negligible.)

(iii) [6 marks]

Compute $x(t)$ for a cell swimming with steady orientation θ_s initially at $x = -h$. Sketch this solution.

5.

a) [3 marks]

Defining H with respect to the pressure, density and velocity of a fluid:

$$H = \frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2.$$

Under what conditions does Bernoulli's theorem, which states that H is a constant throughout the flow field, hold?

b) Consider the 2D flow field around a cylinder of radius a described by the following stream function:

$$\psi = -U(r - \frac{a^2}{r})\sin\theta - \frac{\kappa}{2\pi}\log r, \quad \text{where} \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}.$$

(i) [8 marks]

Using Bernoulli's theorem, compute the pressure distribution outside the cylinder.

(ii) [5 marks]

The total force on the cylinder perpendicular to the free stream velocity is given by

$$F_p = a\int_0^{2\pi} p\sin\theta d\theta.$$

Compute F_p.
(You may use the result that $\int_0^{2\pi}\sin^3\theta d\theta = 0$, $\int_0^{2\pi}\sin^2\theta d\theta = \pi$)

(iii) [4 marks]

Explain how F_p relates to the lift on a wing with reference to the Kutta-Joukowski hypothesis.

6. For the Lighthill elongated body theory for a fish of length l swimming at steady speed U, take $h(x,t)$ as the equation of the centreline, and $w(x,t)$ as the lateral velocity of the water:

$$w(x,t) = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \equiv \mathcal{D}h$$

The rate of increase in kinetic energy of the water surrounding the fish is given by

$$\int_0^l \mathcal{D}(\frac{1}{2}mw^2)dx.$$

(i) [7 marks]

Show that the average total rate of increase of the kinetic energy of the water is given by

$$\frac{1}{2}U[m < w^2 >]_{x=l},$$

stating clearly any assumptions you make. Average is defined in the standard way: $< f >= \frac{1}{T}\int_0^T f(t)dt$, where T is the time period of motion.

The average work done per unit time by body motions is given by

$$< E_L >= [Um < w\frac{\partial h}{\partial t} >]_{x=l}.$$

Consider the following fish undulation (standing wave):

$$h(x,t) = H\sin(kx)\cos(\omega t)$$

(ii) [10 marks]

From energy considerations, show that the mean thrust exerted by the fish on the water is given by

$$< T >= \frac{1}{4}m(l)H^2(w^2 - U^2k^2)\sin^2(kl).$$

You may use the result that $< \cos^2 wt >=< \sin^2 wt >= \frac{1}{2}$.

(iii) [3 marks]

Show that the maximum efficiency is 1/2, where efficiency, η, is defined:

$$\eta = \frac{U < T >}{< E_L >}.$$

7.

a) [3 marks]

Consider the following non-dimensional cell conservation equation for cells swimming in a still fluid:

$$\begin{aligned} \frac{\partial n}{\partial t} &= -\nabla.\mathbf{j}, \\ \text{where the cell flux is given by } \mathbf{j} &= nd\hat{\mathbf{z}} - \nabla n. \end{aligned}$$

Show that the following equilibrium solution satisfies the this governing equation with no flux boundary conditions at $z = 0, z = -1$:

$$n_{equil} = \exp(dz).$$

(b) Consider a small 2D perturbation to this steady equilibrium state:

$$\mathbf{u} = \epsilon\mathbf{u}' = \epsilon(u', 0, w'), \quad n = n_{equil}(z) + \epsilon n'.$$

The non-dimensional linearized Navier Stokes equations are given by:

$$\begin{aligned} \frac{D}{\nu}\frac{\partial \mathbf{u}'}{\partial t} &= -\nabla p_e' - Rn'\hat{\mathbf{z}} + \nabla^2\mathbf{u}', \\ \nabla.\mathbf{u}' &= 0. \end{aligned}$$

(i) [2 marks]

Explain whether you expect bioconvection to occur with reference to the critical Rayleigh number, R_{crit}.

(ii) [15 marks]

Taking the following perturbation

$$w' = \sin(kx)\mathcal{W}(z)e^{\sigma t}, \quad n' = \sin(kx)\mathcal{N}(z)e^{\sigma t}.$$

Compute the 6th order differential equation satisfied by $\mathcal{W}(z)$. You may use the results that for an incompressible flow

$$\begin{aligned} \nabla \wedge (\nabla \wedge \mathbf{u}) &= -\nabla^2\mathbf{u}, \\ \nabla \wedge (\nabla \wedge (n'\hat{\mathbf{z}})) &= -\frac{\partial^2 n'}{\partial x \partial z}\hat{\mathbf{x}} - \frac{\partial^2 n'}{\partial x^2}\hat{\mathbf{z}}, \\ \nabla.(n\mathbf{u}') &= \mathbf{u}'.(\nabla n). \end{aligned}$$