1.

a) [6 marks]

The Navier-Stokes equations of motion can be written as follows:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}.$$

Non-dimensionalise this equation based on characteristic length scale, L, and velocity scale, U, and scale pressure based on the inertial terms. Define an appropriate Reynolds number.

b) A study is performed on the aerodynamics of insect wings. Measurements are made on thin plates shaped exactly like fly wings but with linear dimensions four times larger.

(i) [3 marks]

What should the velocity in the wind tunnel be to correctly model a flow of speed 200cm/s around a real insect wing? State any assumptions you make.

(ii) [6 marks]

The drag on the model wing in the wind tunnel is measured as $3 \times 10^{-3}\text{gcm/s}^2$. Using dimensional arguments, explain why the drag can be written as

$$D = \frac{1}{2}C_D\rho U^2 L^2,$$

where C_D only depends on the Reynolds number and the shape of the wing. Calculate the drag on the real wing explaining carefully your method.

(iii) [5 marks]

How would your answers to (i) and (ii) change if you ran the same experiment in a tank of water? Assume μ_a and ρ_a represent the viscosity and density of air respectively, and μ_w, and ρ_w the corresponding values for water. Express your answer in terms of these variables.

2.

Consider a sphere of radius a moving with velocity $\mathbf{U}$ in unbounded fluid which is at rest at infinity. Take spherical polar coordinates (r, θ, ϕ) with $\theta = 0$ parallel to $\mathbf{U}$ and the origin at the centre of the sphere. Neglecting inertia, the axisymmetric flow generated by the sphere is given by

$$u_r = 2\left(\frac{C}{r} + \frac{D}{r^3}\right)\cos\theta, \quad u_\theta = \left(-\frac{C}{r} + \frac{D}{r^3}\right)\sin\theta, \quad u_\phi = 0.$$

(i) [5 marks]

Verify that this flow is incompressible. You may use the result that

$$\nabla.\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

(ii) [5 marks]

Compute C and D for a rigid sphere of radius a, clearly stating the boundary conditions.

(iii) [5 marks]

The magnitude of the drag force on this sphere is $6\pi a\mu U$. Compute a stokeslet velocity field representing a point force of magnitude F acting at the origin in the direction $\theta = 0$ using the result of (ii).

(iv) [5 marks]

With the assistance of clear diagrams, explain why a stokeslet could be used to model a hovering negatively buoyant copepod. Give an example of one limitation of this model.

3.

Consider a micro-organism which swims at steady velocity $-U\mathbf{i}$ and is propelled by an inextensible flagellum of length L undergoing planar waving motion with wave velocity $V\mathbf{i}$. At $t = 0$ the position of the centreline of the flagellum is given by the parametric equation:

$$\mathbf{R}(s) = (X(s), Y(s)), \quad 0 < s < L,$$

where s is arc length measured from one end.

Resistive force theory states that the force acting on an element of flagellum of length ds moving with velocity $-\mathbf{w}$ is given by

$$F_T = K_T \mathbf{w}.\mathbf{t} ds, \quad F_N = K_N \mathbf{w}.\mathbf{n} ds,$$

where F_T and F_N are the components of force tangential and normal to the flagellum respectively, and $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and normal vectors respectively.

(i) [8 marks]

Show that $-T$, the total force on the flagellum in the $\mathbf{i}$ direction, is given by

$$-T \quad = \quad (K_T - K_N) \int_0^L (\mathbf{w}.\mathbf{t})(\mathbf{t}.\mathbf{i}) ds + K_N \int_0^L \mathbf{w}.\mathbf{i} ds.$$

You can use the identity $\mathbf{w}.\mathbf{i} = (\mathbf{w}.\mathbf{t})(\mathbf{t}.\mathbf{i}) + (\mathbf{w}.\mathbf{n})(\mathbf{n}.\mathbf{i})$.

(ii) [12 marks]

Consider an idealised shape defined by

$$X(s) = \alpha s,$$

where α is some constant such that $0 < \alpha < 1$. The velocity of a material point on the flagellum relative to the fluid far away is $-\mathbf{w}$, where

$$\mathbf{w} = (U - V)\mathbf{i} + \frac{V}{\alpha}\mathbf{t}.$$

Show that the swimming speed for zero thrust swimming (i.e. neglecting the head) is given by

$$U = V \frac{(1 - \alpha^2)(1 - \gamma)}{1 - \alpha^2(1 - \gamma)}, \quad \text{where} \quad \gamma = \frac{K_T}{K_N}.$$

4.

A bottom heavy spherical micro-organism is at position $(x(t), y(t), 0)$ in a planar shear flow, $\mathbf{u} = \gamma y \hat{\mathbf{x}}$. The micro-organism swims at speed v with swimming direction $\mathbf{p} = (\sin\theta, \cos\theta, 0)$ that satisfies the following vector equation:

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}[\hat{\mathbf{y}} - (\hat{\mathbf{y}}.\mathbf{p})\mathbf{p}] + \frac{1}{2}\boldsymbol{\omega} \wedge \mathbf{p},$$

where $\boldsymbol{\omega}$ is the vorticity of the flow.

(i) [11 marks]

Derive a set of differential equations satisfied by $\theta(t), x(t)$ and $y(t)$.

(ii) [3 marks]

Carefully describe the 2 types of motion which depend on the relative magnitudes of B and γ.

(iii) [6 marks]

Compute and sketch the trajectory of the cell swimming with steady orientation θ_s initially at $(0, 0)$. Mark the angle θ_s on the sketch.

5.

(i) [6 marks]

Derive the Bernoulli theorem for steady irrotational flow, starting from Euler's equation of motion for inviscid flow:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p.$$

You can use the identity $(\nabla \wedge \mathbf{u}) \wedge \mathbf{u} = \mathbf{u}.\nabla\mathbf{u} - \nabla(\frac{1}{2}\mathbf{u}^2)$.

(ii) [4 marks]

Consider the 2D irrotational flow given in cylindrical polar coordinates by $\mathbf{u} = \frac{\kappa}{2\pi r}\mathbf{e}_\theta$. Compute the circulation around a circle centred at the origin.

(iii) [5 marks]

Explain how lift is generated by steady flow past a wing, with reference to the Bernoulli theorem and the Kutta-Joukowski hypothesis.

(iv) [5 marks]

In still air, an albatross is observed to steadily glide at an angle α to the horizontal with speed U. The weight of the bird is W. With the aid of force diagrams, what thrust would be required for the bird to fly horizontally with the same speed?

6.

For the Lighthill elongated body theory for a fish of length l swimming at steady speed U, take $h(x,t)$ as the equation of the centreline, and $w(x,t)$ as the lateral velocity of the water:

$$w(x,t) = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \equiv \mathcal{D}h.$$

The work done per unit time by periodic body motions can be written as

$$E_L = \int_0^l \mathcal{D}(mw\frac{\partial h}{\partial t})dx - \int_0^l mw\frac{\partial w}{\partial t}dx,$$

where $m(x)$ is the virtual mass per unit length.

(i) [7 marks]

Show that the average work done per unit time is given by

$$< E_L >= [Um < w\frac{\partial h}{\partial t} >]_{x=l},$$

stating clearly any assumptions you make. Average is defined in the standard way: $< f >= \frac{1}{T}\int_0^T f(t)dt$, where T is the time period of motion.

The average rate of increase in kinetic energy of the water surrounding the fish is given by

$$< \int_0^l \mathcal{D}(\frac{1}{2}mw^2)dx >= [\frac{1}{2}Um < w^2 >]_{x=l},$$

Consider the following fish undulation:

$$h(x,t) = H\sin(\alpha(x - Vt)).$$

(ii) [9 marks]

From energy considerations, show that the mean thrust exerted by the fish on the water is given by

$$< T >= \frac{1}{4}m(l)\alpha^2H^2(V^2 - U^2).$$

You may use the result that $< \cos^2((\alpha(x - Vt)) >= \frac{1}{2}$.

(iii) [4 marks]

If the viscous drag on the fish is βU^2, compute the swimming speed, U, of the fish as a function of $m(l), H$ and V.

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7.

The non-dimensional equations for bioconvection can be written as:

$$\frac{D}{\nu}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p_e - Rn\hat{\mathbf{z}} + \nabla^2\mathbf{u}, \quad (1)$$

$$\nabla.\mathbf{u} = 0, \quad (2)$$

$$\frac{\partial n}{\partial t} = -\nabla.\left(n(\mathbf{u} + d\hat{\mathbf{z}}) - \nabla n\right). \quad (3)$$

(i) [2 marks]

Explain whether you expect bioconvection to occur with reference to the critical Rayleigh number, R_{crit}.

(ii) [3 marks]

Write down suitable boundary conditions for n and $\mathbf{u}$ for rigid boundaries at $z = 0$ and $z = -1$.

(iii) [4 marks]

Show that the following equilibrium solution satisfies the governing equation for cell concentration (equation 3) and boundary conditions.

$$\mathbf{u} = \mathbf{0}, \quad n_{equil} = \exp(dz).$$

(iv) [11 marks]

Consider a small 2D perturbation to equilibrium:

$$\mathbf{u}' = \epsilon\mathbf{u}' = \epsilon\left(\frac{\partial\psi}{\partial z}, 0, -\frac{\partial\psi}{\partial x}\right),$$

$$n' = n_{equil} + \epsilon\mathbf{n}'.$$

Linearise equations (1) and (3) to obtain a coupled pair of equations for $\mathbf{u}'(x, z, t)$ and $n'(x, z, t)$. By eliminating the pressure term, derive a coupled pair of linear differential equations for $\psi(x, z, t)$ and $n'(x, z, t)$. You may use the following results:

$$\nabla \wedge \mathbf{u}' = \nabla^2\psi\hat{\mathbf{y}},$$

$$\nabla \wedge (n'\hat{\mathbf{z}}) = -\frac{\partial n'}{\partial x}\hat{\mathbf{y}},$$

$$\nabla.(n\mathbf{u}') = n(\nabla.\mathbf{u}') + \mathbf{u}'.(\nabla n).$$