1. Define the terms: *strategy; behavioural strategy*, for extensive form games. [3 marks]

In a game for two players, I and II, five cards are placed in a hat, two with 2 on them and three with 0 on them. The players stake is 50p. each. I draws a card from the hat and looks at it without showing II. He can either 'raise', or 'quit'. If he 'raises', he puts 100p. in the kitty; for this he gets 1 point plus the points on the card. If he quits he loses his kitty and the game ends. It is now II's turn: he acts in the same way as I, i.e. draws a card first and, after looking at it, II can either 'raise', or 'quit'. The procedures for 'raising' and 'quitting' are the same as for I, except that he puts 300p. into the kitty and gets 3 points if he 'raises'.

Then it is I's turn again: he can either raise or quit, but does not take a card. To raise he must add 200p. to the kitty, but he gets 2 points in return.

The player with the most points wins the kitty. If the points are equal the kitty is shared. Draw the extensive form for this game. [9 marks]

Give one example of a behavioural strategy, which include every possible action, for both I and II. Work out the payoff when these two strategies are played against each other. [8 marks]

2. Explain what is meant by the terms *subgame, subgame perfect equilibrium,* and *backwards induction.* [5 marks]

In a game for two players, I and II take it in turns to take discs from the table. Players cannot pass. There are m_r red and m_b blue discs on the table. The player who makes the last possible move loses. This game is denoted by (m_r, m_b). On each move, a player can remove one red and one blue disc, or two red discs, or, if $m_r = 0$ or $m_b \leq 1$, one disc of any one colour. Draw the game tree for the games $(2, 2), (3, 2), (2, 3)$ and $(3, 3)$. [6 marks]

Find all the subgame perfect equilibria for each game. Deduce the winners of the games $(4, 3)$ and $(3, 4)$? [9 marks]

3. Explain the terms *belief profile* and *assessment*. [5 marks]

A market trader receives, at £4 each, some goods 40% of which are not of good quality. This number (40%) is known to the buyers. He can tell after inspection which goods are inferior, but after touching them up which costs £3 per item, a buyer cannot tell the difference between superior and inferior goods at the time of buying. The trader can return goods not put on sale to the scrap yard for £3 each, but goods put on sale but not bought cannot be returned and cost the trader all he has spent on them. A buyer, on average, values superior goods at £x each, where x is an unknown quantity at present, and inferior goods at £3, and after a short period after buying a good he can tell the quality of the good. Suppose the trader sells goods at £m each, and puts on sale all superior goods and a proportion p of inferior goods, and buyers, on average, buy a proportion q of goods on sale. Write down the profits per good that the trader and the average buyer gets, per sale. (Assume that the trader has zero profit and his loss is the cost so far, if he does not sell.)

What proportion of inferior goods does the trader try to sell and what proportion of goods does the buyer buy if (i) $x = 9, m = 7$? (ii) $x = 9, m = 6$? Interpret your answers. [15 marks]

4. Explain carefully the terms *possibility sets* for a person and *common knowledge* for a set of people. [5 marks]

A, B, and C are three people who gather in a room. A and C are wearing red hats but B is wearing a blue hat. They know they are all wearing hats, but they do not know the colour of the hat they are wearing and can see each others' headware. Someone tells them that there is at least one person in the room who is wearing a red hat. Say what A's, B's and C's possibility sets are.

A, B and C take it in turns to say whether or not each one can be certain the colour of the hat on their head. Which one is the first to be certain? After A's turn, say what B's possibility sets are, and after A's and B's turns, say what C's possibility sets are. Then justify your previous answer. To know the true state of his hat, what is the minimum amount of information that C needs?

[15 marks]

5. Compare the advantages and disadvantages of the core and the Shapley value as solutions for a cooperative n-person game.

[4 marks]

Four people, A, B, C and D form a committee and have 48, 24, 18 and 10 votes respectively. To ensure the passing of a resolution, m votes are required. Find the minimum winning coalitions when (i) $m = 70$, and (ii) $m = 51$. Treating these situations as simple games, find the core and the Shapley value in each case.

Use these results as illustrations of the advantages and disadvantages you discussed above.

[16 marks]

6. (*i*) A bi-matrix, U_I, U_{II}, for a game for two players, I & II, in an evolutionary model is of the form, where $c > a + d$, and $d > 0$:

$$\begin{pmatrix} a, a & a+d, a+d \\ 0, 0 & a+c+d, a-c+d \end{pmatrix}$$

Show that this game has a mixed strategy solution at $(\mathbf{p}^*, \mathbf{q}^*)$ where $\mathbf{p}^* = (p^*, 1-p^*), \mathbf{q}^* = (q^*, 1-q^*)$ and $p^* = 1 - \frac{d}{c-a}$ and $q^* = \frac{c}{a+c}$.

Show also that any strategy $\mathbf{x}$ for I is such that $\mathbf{x}^t.U_I.\mathbf{q}^*$ is independent of $\mathbf{x}$, and equal to $\frac{a^2+ac+ad}{(a+c)}$.

[10 marks]

(*ii*) In a cooperative TU game for four players, I, II, III, and IV, I & II are identical and III & IV are identical. The coalitional function is:

$$\begin{array}{ccc} v(I,II,III,IV) = 14 & v(I,II,III) = 11 & v(II,III,IV) = 10 \\ v(I,II) = 7 & v(I,III) = 7 & v(III,IV) = 5 \end{array}$$

and all coalitional values for single players are zero.

Given that any extra value is distributed equally between the coalitions at the roots (the STD allocation), find the superplayer solution. Compare your solution with the core solution.

[10 marks]

7. In an evolutionary game for two players, I & II, of the same species with payoff matrix U, what are the conditions for an ESS? Why is a strategy for a Nash equilibrium not always an ESS?

[6 marks]

Suppose there is an evolutionary game, but I belongs to one species and II belongs to a different species, and there are separate payoff matrices U_I, U_{II} for them respectively, as in (i) of question 6:

$$\begin{pmatrix} a, a & a+d, a+d \\ 0, 0 & a+c+d, a-c+d \end{pmatrix}$$

where $c > a+d$, and $d > 0$. Write $(0,1)$ as $\mathbf{y}$. The mixed strategy solution is $(\mathbf{p}^*, \mathbf{q}^*)$ where $\mathbf{p}^* = (p^*, 1-p^*), \mathbf{q}^* = (q^*, 1-q^*)$ and $p^* = 1 - \frac{d}{c-a}$ and $q^* = \frac{c}{a+c}$, (as in question 6); show that

$$\mathbf{y}.(U_I + U_{II}).\mathbf{y} > \mathbf{p}^*.U_I.\mathbf{y} + \mathbf{y}.U_{II}.\mathbf{q}^*.$$

Explain, briefly, why $(\mathbf{p}^*, \mathbf{q}^*)$ is not an ESS?

[14 marks]