Marks will be awarded to the best five answers.
All questions carry equal marks.

1. Explain the term *information set*, and the term *belief* for a player when confronted by an information set. *(3 marks)*

This is a game for two players, I and II. In the first stage each player stakes, in the kitty, £1. Two cards, marked H and L respectively, are placed in a hat, and I draws a card without showing II. After looking at her card, I can either Concede, when the kitty goes to II and the game ends, or Raise by putting an extra £3 into the kitty. Player II then either Concedes, and I gets the kitty and the game ends, or Raises by putting an extra £3 into the kitty. If both players Raise, the card is shown: if the card is H, I wins the kitty: if L the kitty goes to II **AND** the first stage is played again (as a second stage), with H and L again in the hat. The game finishes after this second stage is played, so that if the second card is L, and the actions taken at the second stage are both Raise, I loses *£*8 in total.

Draw the extensive form of this game. *(9 marks)*

Give four examples of pure strategies of player I? How many pure strategies are there? *(3 marks)*

Give, for II, an example of a behavioural strategy, which includes at least one action played with probability different from 0 or 1. Player II believes that player I will Concede at any stage of the game with probability 2/3 if she gets a card with L but will definitely Raise if she gets a H. Work out the expected payoff for II with your behavioural strategy. *(5 marks)*

2. Explain the terms *move, action* and *subgame.* *(3 marks)*

In the game $G[A : (g_0, r_0)]$, two players, A and B, alternately take discs from two piles on a table. One of the piles has green coloured discs and the other red. We designate the numbers in the piles as (g,r), where $g \geq r$; $g_0 \geq r_0$. A goes first and (g_0, r_0) are the initial values of (g, r). The player who moves last loses. At a move, players must, if $r \neq 0$, either remove one red disc or one red and one green disc, and, if $r = 0$, remove one green disc.

Draw the extensive form for the games $G[A : (2, 2)]$ and $G[A : (3, 2)]$, and a schematic extensive form for the game $G[A : (g_0, 2)]$. Show that B can ensure a win in all these games. *(7 marks)*

Using induction, or otherwise, show that the winner of $G[A : (g_0, r_0)]$, if played optimally, depends only on the value of r_0. Deduce that $G[A : (g_0, r_0)]$ can always be won by A if r_0 is odd and by B if r_0 is even. *(10 marks)*

3. Define an *assessment.* (*2 marks*)
A shopkeeper acquires some dresses, $\frac{7}{12}$ of which are good quality and cost him $\pounds 10$ each and $\frac{5}{12}$ of them are poor quality and cost him $\pounds 5$ each. The shopkeeper has to decide whether to sell the dresses in his shop at £30 each or return them to the manufacturers, getting his money back at no cost. If the shopkeeper decides to put the poor quality dresses on sale, he alters them at a cost of $\pounds 10$ each so that, to a buyer, they are superficially indistinguishable from good quality dresses. Buyers value good quality dresses at £40 and poor quality ones at £10, and they realise the quality after wearing the dresses.
Consider this a signalling game: draw its extensive form (the shopkeeper loses his costs if the dresses are not bought.) (*5 marks*)
A buyer knows that the shopkeeper puts all good quality dresses on sale but does not know what percentage p of poor quality dresses are for sale. If she believes $p < 70$ show that she should buy any dress, but if she believes $p > 70$ she should buy no dresses. If $p = 70$ what should she do? (*7 marks*)
What should the shopkeeper do in these three cases? Show that there is an assessment equilibrium when $p = 70$ and the buyer buys dresses with probability 0.5. (*6 marks*)

4. Define the terms *core, Shapley value* for a cooperative n-person game. Discuss their relative merits as solutions. (*4 marks*)

Three sellers I, II, III have identical "antique" chairs, which they are willing to sell at £25, £31 and £36 respectively. A, B, C are three buyers, willing to spend £30, £35 and £37 respectively to buy such a chair.

(i) Consider the situation where only seller I and buyers A and B are involved, as a 3-person cooperative game and write down its characteristic function. Show that its core solution gives I at least $\pounds(25. + x)$ while B pays $\pounds(35. - x)$ for the chair, where $0 \leq x \leq 5$. Does the core solution cover situations where B bribes A not to bid for the chair so that B only pays $\pounds 25$ for the chair? (*3 marks*)

(ii) When all six people are involved, consider it as an ideal market and draw the demand and supply functions. Deduce the number of chairs sold and the limits on the (single) price. (*6 marks*)
Consider this market as a 6-person cooperative game: show that in the core of this game III and A get nothing and that the sum of the payoffs to II and B is £4; find also the limits on the price of chairs. Even though all three buyers are willing to buy at prices above those at which the sellers are willing to sell, why are only two chairs sold in this ideal market? (*7 marks*)

5. What is a dummy voter in a voting game? What is a decisive voting game? In the weighted voting game $\{6 \mid 4, 2, 2, 2, 1\}$ played by five players I,II,III,IV,V respectively, show that V is a dummy voter, but that the game is decisive. *(5 marks)*

Show that the weighted voting games $\{10 \mid 6, 5, 4, 2, 1\}$ and $\{7 \mid 4, 3, 3, 1, 1\}$ are identical. Show also that the game is not decisive. *(6 marks)*

Find the minimal-winning player-sets for the above game, and find the player-sets in which the players are pivotal. Calculate the Shapley-Shubik and the normalised Banzhaf indices for this game. *(9 marks)*

6. What conditions does an Evolutionarily Stable Strategy for a symmetric game with payoff matrix A satisfy? Give the conditions for the ESS when the strategies are given by the value of a continuous variable s and the fitness is $A(s, s^*)$. *(5 marks)*

An important quantity in evolutionary biology is the investment parents make in the rearing of their offspring. We write x_1 and x_2 for the effort a male and a female respectively of a species make in rearing their offspring (in some suitable units). The probability of their offspring reaching maturity is thus $f(x_1 + x_2)$ where f is a monotonically increasing function. A model, due to Parker, of the investment is that parent animals have $X_i, i = 1, 2,$ (where X_i is a constant depending on the species) that can be devoted to producing offspring, so that the expected number of offspring for an adult animal is $X_i f(x_1 + x_2)/(C_i + x_i)$ for $i = 1, 2$. C_i are constants depending on the species. Show that at the ESS:

$$C_1 + x_1^* = C_2 + x_2^*, \tag{1}$$

If $f(x) = 1 - 1/x, x > 1$ deduce the values of x_1^* and x_2^*. *(15 marks)*

7. Discuss three of the following topics in game theory, giving examples and mathematical reasoning if possible and relevant:

(*i*) *threats, promises and credibility*:

(*ii*) *Harsanyi's theory for games of incomplete information*:

(*iii*) *coalitions, including the ideas of partitions, and superplayers*:

(*iv*) *the replicator equation and its justification.*