PAPER CODE NO. MATH427	EXAMINER:	
	DEPARTMENT:	TEL. NO:

THE UNIVERSITY of LIVERPOOL

SUMMER 2006 EXAMINATIONS

Master of Mathematics: Year 4

WAVES. MATHEMATICAL MODELLING

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. Your best answers to FOUR questions only will be taken into account.

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1. **i.** Find the characteristics of the Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the lower half plane $y < 0$, and hence put it into its canonical form.

[9 marks]

ii. **(a)** The continuity equation and Euler's equations of motion (no body force) are linearised by assuming that $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$ and $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where the terms with the indices 0 are constants and the terms with the indices 1 denote small perturbations to the respective fields, to obtain

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 + \mathbf{u}_0 \cdot \nabla \rho_1 = 0, \qquad \frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + \frac{c_0^2}{\rho_0}\nabla \rho_1 = 0.$$

Here, ρ, p and $\mathbf{u}$ stand for density, pressure and velocity of the medium, respectively; $c_0^2 = \gamma\, p_0/\rho_0$ is the leading order wave speed of the medium, where $\gamma > 1$ is a dimensionless constant.

Now, assuming that the total velocity field is derived from a potential ϕ, show that

$$\rho_1 = -\frac{\rho_0}{c_0^2}(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla)\phi, \quad p_1 = -\rho_0(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla)\phi,$$

and

$$(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla)^2\phi - c_0^2 \nabla^2 \phi = 0.$$

[11 marks]

(b) Describe appropriate boundary conditions in terms of the velocity potential ϕ for the following boundaries:

- A fixed rigid boundary,
- A traction free boundary,
- An interface boundary between two fluids, assuming that the velocity field in the second medium is derived from a potential ϕ^*.

[5 marks]

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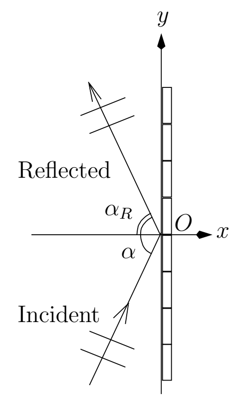
2. **i.** A time-harmonic plane wave characterised by a complex velocity potential $\psi^{(I)}$ with amplitude A, radian frequency ω and speed c is incident to a wall covered with acoustic tiles as shown in the figure, generating a reflected plane wave. The $x-$axis is chosen to be horizontal and the $y-$axis vertically upwards. α denotes the angle of the incident wave, whereas α_R stands for the angle of the reflected wave, both indicated clearly in the diagram.



Write down the complex velocity potential $\psi^{(I)}$ taking into account the direction of propagation of the wave.

[3 marks]

The acoustic tiles are assumed to absorb energy. Thus the boundary condition on the tiled wall is given as

$$\frac{\partial \psi^{(total)}}{\partial x} = -\lambda \frac{\partial \psi^{(total)}}{\partial t},$$

where λ is the wavelength and $\psi^{(total)}$ is the combined fields of the incident and reflected waves.

Show that $\alpha = \alpha_R$ and hence deduce that the complex velocity potential for the reflected wave is given by

$$\psi^{(R)}(x, y, t) = A \frac{\cos(\alpha) - \lambda c}{\cos(\alpha) + \lambda c} \, \mathrm{e}^{-i\omega\{t - \frac{1}{c}[-x\cos(\alpha) + y\sin(\alpha)]\}}.$$

[10 marks]

Introducing the parameter $s = \sin(\alpha)$ and formally assuming that $s > 1$, use the complex potential above to show that an evanescent wave, i.e. a wave, in this particular case, which is oscillatory in $y-$direction and which decays as $x \to -\infty$, can be produced.

[6 marks]

ii. Solve the following nonlinear initial value problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad x \in (0, \infty),\ t > 0,$$
$$u(x, 0) = a\,x, \quad t > 0,$$

where $a \in \mathbf{R}$ is constant.

[6 marks]

3. **i.** **(a)** Show that the three-dimensional wave equation posed for the velocity potential ϕ, i.e.

$$\phi_{tt} - c^2 \nabla^2 \phi = 0,$$

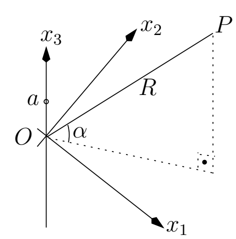
can be reduced to

$$(r\phi)_{tt} - c^2 \frac{\partial^2}{\partial r^2}(r\phi) = 0,$$

and thus possesses a solution of the form $F(t - r/c)/r$, characterising waves propagating radially outwards from a source, where $r = |\mathbf{x}|$.

[6 marks]

(b) A simple source is placed at $(0, 0, a)$ above a fixed rigid wall $x_3 = 0$, as shown in the diagram. If the velocity potential for the incident wave characterising an outgoing wave from this source is given as



$$\Phi^I = -\frac{1}{r_1} S^I(t - r_1/c), \quad r_1 = [x_1^2 + x_2^2 + (x_3 - a)^2]^{1/2},$$

where S^I denotes the strength of the source, find the velocity potential for the reflected wave, by using a fictitious image source.

Now, assuming that the strength of the source is of the form $S^I(t) = \mathrm{e}^{-i\omega t}$, show that the total velocity potential at point P can be approximated by

$$\Phi^{\text{total}}\big|_P \sim -\frac{2}{R} \cos[ka \sin(\alpha)]\, \mathrm{e}^{i(kR - \omega t)},$$

under the assumption that $a \ll R$, where R is the distance from O to P, and α is the angle between the fixed $x_1 x_2-$plane and OP.

[12 marks]

ii. Solve the following nonlinear initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in (0, \infty),\ t > 0,$$

$$u(x,0) = \begin{cases} 2, & -\infty < x \le -1, \\ 2x^2, & -1 \le x \le 0, \\ 1, & 0 \le x < \infty. \end{cases}$$

[7 marks]

4. Consider the following first-order quasilinear hyperbolic system

$$\mathbf{v}_t + A\,\mathbf{v}_x = \mathbf{0},$$

where

$$\mathbf{v} = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad A = \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}.$$

Here, ρ denotes the density, u the velocity and c the wave speed of a flow.

(a) Write down the differential equations defining the characteristics for this system.

[8 marks]

(b) Prove that, if we can find a function Γ (Riemann invariant) associated with an eigenvalue λ, satisfying

$$A^{\top} \frac{\partial \Gamma}{\partial \mathbf{v}} = \lambda \frac{\partial \Gamma}{\partial \mathbf{v}},$$

then Γ is constant along the corresponding characteristic.

[10 marks]

(c) Hence using

$$c(\rho) = (\gamma B \rho^{\gamma-1})^{1/2},$$

where $\gamma > 1$, $B > 0$ are constants, deduce that

$$\Gamma_{\pm} = u \pm \frac{2c}{\gamma - 1}$$

are invariant along characteristics.

[7 marks]

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5. **(a)** Give the definition of a weak solution of the following partial differential equation

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{a}(\mathbf{v})}{\partial x} = \mathbf{0}.$$

[3 marks]

(b) Give the definition of a shock trajectory Γ. Show that the jump across Γ associated with $\mathbf{a}(\mathbf{v})$ is equal to zero, provided that Γ is parallel to the $t-$axis. [12 marks]

(c) Find a weak solution to the following Cauchy problem

$$2\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} = 0,$$

$$u(x,0) = \begin{cases} 2, & 0 < x, \\ 0, & x > 0. \end{cases}$$

[10 marks]

6. The linearised problem for water waves in a channel of width a along the x_2-direction, is given in terms of a complex velocity potential ψ as

$$\nabla^2 \psi = 0, \quad x_3 \in (0,h),$$

$$\mathbf{n} \cdot \nabla \psi|_{x_3=0} = 0, \qquad (\frac{\partial^2 \psi}{\partial t^2} + g\frac{\partial \psi}{\partial x_3})|_{x_3=h} = 0,$$

where $\mathbf{n}$ is the unit vector normal to the bottom of the channel, g is the gravitational acceleration and h is the unperturbed height of the water.

Show that the dispersion relation

$$\omega^2 = gk\tanh(kh),$$

between the radian frequency ω and the wave number k, holds. Discuss the cases when $h \to 0$ and $h \to \infty$.

[15 marks]

Also, show that the velocity components are given by

$$u_1 = 0,\ u_2 = Ck\cosh(kx_3)\sin(\omega t - kx_2),\ u_3 = Ck\sinh(kx_3)\cos(\omega t - kx_2),$$

and hence, stating the shape of their trajectories, find the particle paths. How do the trajectories change when $x_3 \to 0$ and $x_3 \to h$?

[10 marks]