THE UNIVERSITY of LIVERPOOL

# SUMMER 2007 EXAMINATIONS

## MATHEMATICAL BIOLOGY

TIME ALLOWED: Two Hours and a Half

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Full marks can be obtained for complete answers to FIVE questions.
Only the best FIVE answers will be counted.

Throughout this paper standard notation is used. Thus $X$, $Y$ and $Z$ denote population densities of susceptible, infected and removed or immune individuals, respectively. Additionally, $N$ or $H$ is the total density of host individuals. Furthermore, $\beta$ is the transmission parameter, $\gamma$ is the rate of recovery, $\mu$ is the death rate, $r$ is the intrinsic growth rate, $\alpha$ is the pathogenicity, $\Gamma$ is the rate of exiting the infected class, $\lambda$ is the force of infection and $D$ is the diffusion constant.

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1. The dynamics of an epidemic without removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY,$$

where, initially, there are $\frac{4}{5}N$ susceptibles and $\frac{1}{5}N$ infectives.

(i) Find the number of susceptibles as a function of time. [8 marks]

(ii) Find the equation of the epidemic curve and locate and evaluate its maximum. [5 marks]

(iii) Express the equation of the epidemic curve in a simplified form, which uses the time at the maximum as origin and comment very briefly on your result. [7 marks]

2. The dynamics of an epidemic with removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY - \gamma Y,$$
$$\frac{dZ}{dt} = \gamma Y.$$

Assume that initially there are no removed individuals and that the number of infecteds is very small.

(i) Show that the peak number of deaths per unit time occurs when $X = X_T$ where $X_T = \gamma / \beta$ .

[4 marks]

(ii) Find the values of $Y$ and $Z$ at the peak as a fraction of $X_0$ the initial number of susceptibles.

Express these quantities in terms of the basic reproduction ratio $R_0$. [8 marks]

(iii) Derive the transcendental equation

$$N - X_0 \exp(-Z / X_T) - Z = 0$$

for the number of individuals removed ($Z$) as $t \to \pm\infty$ . [3 marks]

(iv) Define the intensity $i$ of the epidemic and obtain the final size equation in terms $i$ and $R_0$.

[5 marks]

3. With births and deaths included, the dynamics of the spread of a typical childhood infection in an unvaccinated population are given by the equations

$$\frac{dX}{dt} = \varepsilon N - \varepsilon X - R_0(1+\varepsilon)XY/N,$$
$$\frac{dY}{dt} = R_0(1+\varepsilon)XY/N - (1+\varepsilon)Y,$$
$$\frac{dZ}{dt} = Y - \varepsilon Z.$$

Here, time is measured in units of the mean time for recovery and $R_0$ is the basic reproduction ratio.

(i) Explain why $\varepsilon$ is a small parameter. [2 marks]

(ii) Find the equilibrium states and analyse these for feasibility and stability. [12 marks]

(iii) Find an expression for the period of the inter-epidemic oscillations in the endemic state. Explain any approximations that you make. [6 marks]

4. (i) Define the term *basic reproduction ratio* $R_0$ used in epidemiology. [2 marks]

(ii) Define the next generation matrix for a general multi-group model. Show that the dominant eigenvalue of this matrix can be identified as the basic reproduction ratio. [5 marks]

(iii) Write down a model for the dynamics of malaria explaining the meaning of the terms you introduce. [4 marks]

(iv) Derive $R_0$ for your model of malaria directly from the definition of (i). [5 marks]

(v) For your model of malaria, obtain the next generation matrix and hence find the basic reproduction ratio. [3 marks]

(vi) Comment briefly on the outcomes of your two calculations of $R_0$ for malaria. [1 marks]

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5. (i) Derive the equations

$$\frac{\partial X}{\partial t}+\frac{\partial X}{\partial a}=-\lambda(a,t)X(a,t)-\mu(a)X(a,t),$$

$$\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial a}=\lambda(a,t)X(a,t)-\Gamma(a)Y(a,t),$$

$$\frac{\partial Z}{\partial t}+\frac{\partial Z}{\partial a}=\gamma(a)Y(a,t)-\mu(a)Z(a,t),$$

for an age-dependent epidemic process. Explain the meaning of each quantity appearing in these equations. [10 marks]

(ii) Deduce the equations for the dynamics of the total population in the case of age-independent parameters. [4 marks]

(iii) Write down the equations for the age distributions in the endemic state and, assuming only that the disease induced death rate is zero, derive formulas for the endemic age-distribution of the fraction of hosts susceptible. [6 marks]

6. The dynamics of a host-parasite association in which the parasite affects host numbers are given by

$$\frac{dH}{dt}=rH-qH^2-\alpha Y,$$

$$\frac{dY}{dt}=\beta XY-\Gamma Y.$$

(i) Describe, very briefly, the assumptions made when such a model is used. [3 marks]

(ii) Find the uninfected equilibrium states of this model. [2 marks]

(iii) Find the value of the susceptible population $X$ in the infected equilibrium state and determine a quadratic equation satisfied by the total population $H$ in this state. [4 marks]

(iv) Find conditions in simplified form for the feasibility and stability of all the equilibria and comment very briefly on your results. [11 marks]

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7. A spatio-temporal epidemic model is given by the equations

$$\frac{\partial X}{\partial t} = -\beta XY + D\frac{\partial^2 X}{\partial x^2},$$
$$\frac{\partial Y}{\partial t} = \beta XY - \alpha Y + D\frac{\partial^2 Y}{\partial x^2}.$$

(i) Show how to write this in the following non-dimensional form

$$\frac{\partial X}{\partial t} = -XY + \frac{\partial^2 X}{\partial x^2},$$
$$\frac{\partial Y}{\partial t} = XY - \rho Y + \frac{\partial^2 Y}{\partial x^2}.$$

This non-dimensional form is used in the remainder of the question. [6 marks]

(ii) Find the differential equations satisfied by travelling wave solutions $X(z), \ Y(z)$ where $z = x - ct$.

[3 marks]

(iii) State the conditions to be imposed on $X(z), \ Y(z)$ as $z \to \pm\infty$. Explain the origin of these conditions. [2 marks]

(iv) Linearize the equations near the leading edge of the wave and derive the minimum wave speed and a threshold condition, involving the basic reproduction ratio, for the propagation of the epidemic wave. [9 marks]