MATH426

MAY 2001

Full marks can be obtained for complete answers to FIVE questions.

Only the best FIVE answers will be counted.

Throughout this paper standard notation is used. Thus *X*, *Y* and *Z* denote population densities of susceptible, infected and removed or immune individuals, respectively. Additionally, *N* or *H* is the total density of host individuals. Furthermore, β is the transmission parameter, γ is the rate of recovery, ν is the rate of loss of immunity, μ is the death rate, *r* is the intrinsic growth rate, α is the pathogenicity, Γ is the rate of exiting the infected class, λ is the force of infection and *D* is the diffusion constant.

1. The dynamics of an epidemic without removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY,$$

where, initially, there are κN susceptibles and $(1 - \kappa)N$ infectives with κ between 0 and 1.

Find the number of susceptibles as a function of time. Find also the equation of the epidemic curve and locate and evaluate its maximum. [20 marks]

2. The dynamics of an epidemic with removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY - \gamma Y,$$
$$\frac{dZ}{dt} = \gamma Y.$$

Show that the peak of the epidemic occurs when $X = X_T = \gamma / \beta$. Take the origin of time to be at the peak of the epidemic and hence show that

$$t = \frac{1}{\gamma} \int_0^{Z-Z_0} \frac{dW}{\left\{Y_0 - W + X_T\left(1 - e^{-W/X_T}\right)\right\}}.$$

Explain the notation. Derive a transcendental equation for the values of Z approached as $t \to \pm\infty$ and show by approximating the denominator of the integrand above which is which. Illustrate your answer graphically and determine the values of Y in the same limit. Define the intensity of the epidemic and obtain the final size equation. [20 marks]

3. Define the term *basic reproduction ratio* R_0 used in epidemiology. Derive R_0 in the case of a population of size *N*, with transmission parameter β and per capita rate Γ for leaving the infective state. Include a derivation of the average time in this state.

Define the next generation matrix for a general multi-group model. Derive the long-term behaviour characterised by this matrix and identify the basic reproduction ratio.

A certain criss-cross venereal infection model has dynamics given by

$$\frac{dX_1}{dt} = -\beta_{12} X_1 Y_2, \qquad \frac{dX_2}{dt} = -\beta_{21} X_2 Y_1,$$
$$\frac{dY_1}{dt} = \beta_{12} X_1 Y_2 - \gamma_1 Y_1, \qquad \frac{dY_2}{dt} = \beta_{21} X_2 Y_1 - \gamma_2 Y_2,$$
$$\frac{dZ_1}{dt} = \gamma_1 Y_1, \qquad \frac{dZ_2}{dt} = \gamma_2 Y_2.$$

Describe, very briefly, the assumptions made when such a model is used. Obtain the next generation matrix and the basic reproduction ratio. Derive a threshold condition involving population sizes for the growth of infection. [20 marks]

4. With births and deaths included, the dynamics of an epidemic are given by the equations

$$\frac{dX}{dt} = \mu N + \nu Z - \mu X - \beta XY,$$
$$\frac{dY}{dt} = \beta XY - (\gamma + \mu)Y,$$
$$\frac{dZ}{dt} = \gamma Y - (\mu + \nu)Z.$$

(Notice the term modelling loss of immunity.)

These equations also describe endemic behaviour. Find the equilibrium states and analyse these for feasibility and stability. Find, in the case $\nu = 0$, an expression for the period of the inter-epidemic oscillations in the endemic state. Explain any assumptions you make. [20 marks]

5. Derive the equations

$$\frac{\partial X}{\partial t}+\frac{\partial X}{\partial a}=-\lambda(a,t)X(a,t)-\mu(a)X(a,t),$$

$$\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial a}=\lambda(a,t)X(a,t)-\Gamma(a)Y(a,t),$$

$$\frac{\partial Z}{\partial t}+\frac{\partial Z}{\partial a}=\gamma(a)Y(a,t)-\mu(a)Z(a,t),$$

for an age-dependent epidemic process. Explain the meaning of each quantity appearing in these equations.

Deduce the equations for the dynamics of the total population in the case of age-independent parameters.

Write down the equations for the age distributions in the endemic state and, assuming only that the disease induced death rate is zero, derive formulas for the endemic age-distribution of the fraction of hosts susceptible. [20 marks]

6. The basic dynamics of host-parasite associations in which the parasite affects host numbers are given by

$$\frac{dH}{dt}=rH-\alpha Y,$$

$$\frac{dY}{dt}=\beta XY-\Gamma Y.$$

Describe, very briefly, the assumptions made when such a model is used. Explain how the equations would need to be modified to include the effect of transmission by free-living infective stages.

In the basic model, investigate the possibility of the host population growing at a reduced exponential rate ρ and hence determine when the pathogen is not able to regulate the host. Determine this ρ and the corresponding behaviour of X. Provide a similar analysis in the case of transmission by free-living infective stages. (You are not required to discuss the feasibility of your results.) [20 marks]

7. A spatio-temporal epidemic model is given by the equations

$$X_t = -\beta XY,$$
$$Y_t = \beta XY - \alpha Y + DY_{xx}.$$

Explain the significance of each of the terms in these equations.

Obtain a non-dimensionalised version of these equations. Derive the corresponding differential equations satisfied by travelling waves. Linearise these near the leading edge of the wave and hence derive a condition for the propagation of such a wave and a condition on the wave speed. How would this condition be interpreted in a spatially uniform situation? What speed would you expect the wave actually to have? [20 marks]