MATH426

MAY 2000

Full marks can be obtained for complete answers to FIVE questions.

Only the best FIVE answers will be counted.

Throughout this paper standard notation is used. Thus X, Y and Z denote population densities of susceptible, infected and immune individuals, respectively. Additionally, N or H is the total density of host individuals. Furthermore, β is the transmission parameter, γ is the rate of recovery, ν is the rate of loss of immunity, μ or b is the death rate, p is a vaccination parameter, r is the intrinsic growth rate, α is the pathogenicity, λ is the force of infection and D is the diffusion constant.

1. The dynamics of an epidemic without removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY,$$

where, initially, there are 2 infectives and $N-2$ susceptibles.

Find the number of susceptibles as a function of time. Find also the equation of the epidemic curve and locate and evaluate its maximum.

2. With births, deaths and a vaccination protocol included, the dynamics of an epidemic are given by the equations

$$\frac{dX}{dt} = \mu N(1-p) + \nu Z - \mu X - \beta XY,$$
$$\frac{dY}{dt} = \beta XY - (\gamma + \mu)Y,$$
$$\frac{dZ}{dt} = \mu Np + \gamma Y - (\mu + \nu)Z.$$

(Notice the term modelling loss of immunity.)

These equations also describe endemic behaviour. Find the equilibrium states and analyse these for feasibility and stability. Determine a threshold condition, involving p, for the eradication of the infection in the long-term.

3. Derive the equations

$$\frac{\partial X}{\partial t}+\frac{\partial X}{\partial a}=-\lambda(a,t)X(a,t)-\mu(a)X(a,t),$$

$$\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial a}=\lambda(a,t)X(a,t)-\Gamma(a)Y(a,t),$$

$$\frac{\partial Z}{\partial t}+\frac{\partial Z}{\partial a}=\gamma(a)Y(a,t)-\mu(a)Z(a,t),$$

for an age-dependent epidemic process. Explain the meaning of each quantity appearing in these equations.

Write down the equations for the age distributions in the endemic state and, assuming only that the disease induced death rate is zero, derive a formula for the fraction of hosts susceptible in this state. Explain how to calculate the average age at infection in the endemic state and, assuming that λ is age independent, calculate this in the two cases:

(i) μ is a constant independent of age,

(ii) $\mu(a)=0$ for $a<L$, $\mu(a)=\infty$ for $a>L$.

Very briefly, compare your results.

4. Define the next generation matrix for a <u>general</u> multi-group model. Derive the long-term behaviour characterised by this matrix and identify the basic reproduction ratio.

A certain criss-cross venereal infection model has dynamics given by

$$\frac{dX_1}{dt} = -\beta_{12} X_1 Y_2, \qquad \frac{dX_2}{dt} = -\beta_{21} X_2 Y_1,$$
$$\frac{dY_1}{dt} = \beta_{12} X_1 Y_2 - \gamma_1 Y_1, \qquad \frac{dY_2}{dt} = \beta_{21} X_2 Y_1 - \gamma_2 Y_2,$$
$$\frac{dZ_1}{dt} = \gamma_1 Y_1, \qquad \frac{dZ_2}{dt} = \gamma_2 Y_2.$$

Describe, very briefly, the assumptions made when such a model is used. Obtain the next generation matrix and the basic reproduction ratio. Derive a threshold condition involving population sizes for the growth of infection.

A more general two-group model has dynamics given by

$$\frac{dX_1}{dt} = -\beta_{11} X_1 Y_1 - \beta_{12} X_1 Y_2, \qquad \frac{dX_2}{dt} = -\beta_{21} X_2 Y_1 - \beta_{22} X_2 Y_2,$$
$$\frac{dY_1}{dt} = \beta_{11} X_1 Y_1 + \beta_{12} X_1 Y_2 - \gamma_1 Y_1, \qquad \frac{dY_2}{dt} = \beta_{21} X_2 Y_1 + \beta_{22} X_2 Y_2 - \gamma_2 Y_2,$$
$$\frac{dZ_1}{dt} = \gamma_1 Y_1, \qquad \frac{dZ_2}{dt} = \gamma_2 Y_2.$$

Obtain the next generation matrix. Derive the threshold value of $\beta_{12}\beta_{21}/\beta_{11}\beta_{22}$ involved in determining whether or not there is growth of infection.

5. The dynamics of host-parasite associations in which the parasite affects host numbers and vertical transmission of infection occurs are given by

$$\frac{dH}{dt} = rH - \alpha Y,$$
$$\frac{dY}{dt} = \beta XY - (\alpha + b + \gamma - aq)Y.$$

Here, a fraction q of the births are of infected young – the quantity a being the birth rate.

The pathogen is unable to regulate the host if $\alpha < r$. Show that, when this inequality holds, the host can grow exponentially at a rate which is less than that in the absence of the parasite. Determine this rate ρ and the corresponding behaviour of X.

In the case where the host is regulated, find the equilibrium level of H and of the prevalence y. Find when this state is feasible and stable. Identify, very briefly, one consequence of increasing q.

6. A spatio-temporal epidemic model is given by the equations

$$\frac{\partial X}{\partial t} = -\beta XY + D\frac{\partial^2 X}{\partial x^2},$$
$$\frac{\partial Y}{\partial t} = \beta XY - \alpha Y + D\frac{\partial^2 Y}{\partial x^2}.$$

Show how to write this in the following non-dimensional form

$$\frac{\partial X}{\partial t} = -XY + \frac{\partial^2 X}{\partial x^2},$$
$$\frac{\partial Y}{\partial t} = XY - \rho Y + \frac{\partial^2 Y}{\partial x^2}.$$

This non-dimensional form is used in the remainder of the question.

Find the differential equations satisfied by travelling wave solutions $X(z)$, $Y(z)$ where $z = x - ct$.

Explain why the conditions $Y(\infty) = Y(-\infty) = 0$ and $0 \le X(-\infty) < X(\infty) = 1$ are imposed here.

Derive the minimum wave speed c and a threshold condition, involving the basic reproduction ratio, for the propagation of an epidemic wave.

7. A spatio-temporal model for the spread of rabies which includes births and deaths of the host takes the non-dimensional form

$$\frac{\partial X}{\partial t} = X(1 - X - Y),$$
$$\frac{\partial Y}{\partial t} = aY(X - b) + \frac{\partial^2 Y}{\partial x^2},$$

where a and b are unspecified parameters.

Obtain the ordinary differential equations

$$cX' = X(1 - X - Y),$$
$$cY' = aY(X - b) + Y'',$$

where the prime denotes differentiation with respect to $z = x + ct$, for a constant shape travelling wave solution of the model.

Write these ordinary differential equations as a first-order system. Find the equilibrium points of this system. Analyse the equilibrium which corresponds to a uniform uninfected population for stability. Find the minimum possible speed of a travelling wave corresponding to the spread of infection. Comment on the feasibility of the equilibrium which corresponds to a uniform infected population and find the characteristic equation of the corresponding Jacobian. Discuss its eigenvalues in the cases where a is zero and where a is small and positive. What does this tell you about the nature of the travelling wave?