PAPER CODE NO.
MATH421

THE UNIVERSITY *of* LIVERPOOL

JANUARY 2008 EXAMINATIONS

MMath: Year 4

LINEAR DIFFERENTIAL OPERATORS IN MATHEMATICAL PHYSICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to FIVE questions. Only the best 5 answers will be taken into account.

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1. Let $D = \{\mathbf{x} : |\mathbf{x}| < 1\}$ be a unit disk with the centre at the origin, and let the energy functional $\mathcal{E}$ in D be defined by

$$\mathcal{E}(u; D) = \int_D |\nabla u(\mathbf{x})|^2 d\mathbf{x}. \tag{1}$$

The class $\mathcal{A}$ of admissible functions is introduced as follows

$$\mathcal{A} = \{\text{smooth } \ u \text{ on } D : \quad \mathcal{E}(u; D) < +\infty, u = \cos(2\theta) \ \text{ when } |\mathbf{x}| = 1\}, \tag{2}$$

where θ is the polar angle.

Prove that

$$\mathcal{E}(x_1^2 - x_2^2; D) \leq \mathcal{E}(u; D) \ \text{ for any } \ u \in \mathcal{A}. \tag{3}$$

[20 marks]

Hint: Verify that $x_1^2 - x_2^2$ is harmonic in D and $x_1^2 - x_2^2 \in \mathcal{A}$.

2. Let $w(\mathbf{x}, t)$ satisfy the wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \ \text{ for } \ x \in (-1, 1), \ t > 0, \tag{4}$$

together with the initial and boundary conditions

$$w(x, 0) = \cos(\pi x), \ \frac{\partial w}{\partial t}(x, 0) = 0 \ \text{ for } \ x \in (-1, 1), \tag{5}$$

$$w(-1, t) = \frac{\partial w}{\partial t}(1, t) = 0 \ \text{ for all } t > 0. \tag{6}$$

Given the energy functional $E(w, t)$ defined by

$$E(w,t) = \int_{-1}^{1} \Big\{ (\frac{\partial w}{\partial t}(x,t))^2 + (\frac{\partial w}{\partial x}(x,t))^2 \Big\} dx, \tag{7}$$

evaluate $E(w, 5)$.

[20 marks]

Hint: Verify that the solution $w(x, t)$ of the problem (4)–(6) satisfies the identity $E(w, t_1) = E(w, t_2)$ for any $t_1, t_2 \geq 0$.

3. This question uses the notion of distributions.

(a) Let

$$f(x) = \begin{cases} 1, & x > 3 \\ (x-2), & 2 < x \leq 3 \\ 0, & x \leq 2 \end{cases} \tag{8}$$

Interpret $f''(x)$ in the sense of distributions and verify that

$$(f''(x), \varphi(x)) = \varphi(2) - \varphi(3), \tag{9}$$

for any test function $\varphi(x) \in \mathcal{D}(\mathbf{R})$.

[7 marks]

(b) Let $f(\mathbf{x}) = \log |\mathbf{x}|$, where $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$. Evaluate

$$\Delta f(\mathbf{x}) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \tag{10}$$

in the sense of distributions and prove that

$$\Delta(\log |\mathbf{x}|) = 2\pi\delta(\mathbf{x}), \tag{11}$$

where $\delta(\mathbf{x})$ is the Dirac delta function.

[13 marks]

4. This question requires analysis of solutions of the wave equation.

(a) The fundamental solution $G(x,t)$ of the wave equation satisfies the following Cauchy problem

$$\frac{\partial^2 G(x,t)}{\partial x^2} - \frac{\partial^2 G(x,t)}{\partial t^2} = 0, \;\; x \in (-\infty, +\infty), t > 0, \tag{12}$$

$$G(x,0) = 0, \;\; \frac{\partial G}{\partial t}(x,0) = \delta(x), \;\; x \in (-\infty, +\infty). \tag{13}$$

Verify that $G(x,t)$ can be represented in the form

$$G(x,t) = \frac{1}{2}(H(x+t) - H(x-t)). \tag{14}$$

[10 marks]

(b) Find the solution $u(x,t)$ of the Cauchy problem for an infinite vibrating string

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0, \ x \in (-\infty, +\infty), t > 0, \qquad (15)$$

with the initial conditions

$$u(x,0) = 0, \ \frac{\partial u}{\partial t}(x,0) = e^{-|x|}, \quad x \in (-\infty, \infty). \qquad (16)$$

[10 marks]

Hint: Use the D'Alembert formula.

5. Boundary value problems for Laplace's operator are included in this question.

(a) Derive the representation of Green's function $G(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, for the operator $-\Delta$ in the half-plane

$$\mathbf{R}^2_+ = \{(x_1, x_2) : x_1 \in (-\infty, \infty), \ x_2 > 0\}. \qquad (17)$$

Green's function is defined as the solution of the following boundary value problem

$$\frac{\partial^2 G}{\partial x_1^2}(\mathbf{x}, \mathbf{y}) + \frac{\partial^2 G}{\partial x_2^2}(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^2_+, \qquad (18)$$

with the Dirichlet boundary condition

$$G(\mathbf{x}, \mathbf{y}) = 0 \ \text{ when } \ x_1 \in (-\infty, \infty), x_2 = 0, \mathbf{y} \in \mathbf{R}^2_+. \qquad (19)$$

Here $\delta(\mathbf{x})$ is the Dirac delta function.

[10 marks]

(b) Find a solution of the following Dirichlet boundary value problem in the half-plane

$$\frac{\partial^2 u}{\partial x_1^2}(x_1, x_2) + \frac{\partial^2 u}{\partial x_2^2}(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathbf{R}^2_+, \qquad (20)$$

$$u(x_1, 0) = \delta(x_1 - 1) - \delta(x_1 + 1), \qquad (21)$$

$$\lim_{x_1^2 + x_2^2 \to \infty} u(x_1, x_2) = 0. \qquad (22)$$

[10 marks]

6. Let (r,θ,φ) be spherical coordinates. A function $u(r,\theta,\varphi)$ harmonic in the interior of the unit sphere satisfies the integral Poisson's formula

$$u(r,\theta,\varphi)=\frac{1-r^2}{4\pi}\int_0^{\pi}d\theta'\int_0^{2\pi}d\varphi'\frac{u(1,\theta',\varphi')\sin\theta'}{(1+r^2-2r\cos\gamma)^{3/2}}, \tag{23}$$

where

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi'-\varphi),\quad 0\le r<1.$$

Derive the integral representation for a function $v(r,\theta,\varphi)$, which satisfies Laplace's equation in the exterior of the unit sphere ($r>1$), and the boundary condition $v(1,\theta,\varphi)=\Psi(\theta,\varphi)$, where $\Psi(\theta,\varphi)$ is a given smooth function.

Write down the representation of v for the case when $\Psi\equiv 1$.

[20 marks]

Hint: Use Kelvin's inversion for harmonic functions.

7. This question involves solutions of the heat equation.

(a) Verify that the function

$$G(x,t)=\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}, \tag{24}$$

satisfies the heat equation

$$\frac{\partial G(x,t)}{\partial t}=\frac{\partial^2 G(x,t)}{\partial x^2}\quad\text{for } x\in(-\infty,\infty),\ t>0, \tag{25}$$

and the initial condition

$$G(x,0)=\delta(x),\quad x\in(-\infty,\infty), \tag{26}$$

where $\delta(x)$ is the Dirac delta function.

[10 marks]

(b) Using the notion of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha,$$

determine the solution of the following Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad x \in (-\infty, \infty), \ t > 0, \tag{27}$$

$$u(x,0) = \begin{cases} -2, & \text{when } x < 1, \\ 3, & \text{when } x \geq 1. \end{cases} \tag{28}$$

[10 marks]