PAPER CODE NO.
MATH421

THE UNIVERSITY
of LIVERPOOL

JANUARY 2006 EXAMINATIONS

MMath: Year 4

LINEAR DIFFERENTIAL OPERATORS IN MATHEMATICAL PHYSICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to FIVE questions. Only the best 5 answers will be taken into account.

1. Let $u(x,t)$ be a solution of the following initial boundary value problem for the wave equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) - 4\frac{\partial^2 u}{\partial t^2}(x,t) = 0, \quad x > 0,\ t > 0, \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0,\ t > 0, \tag{2}$$

and the initial conditions

$$\frac{\partial u}{\partial t}(0,t) = 0,\ x > 0, \tag{3}$$

$$u(x,0) = \begin{cases} -|x-3|+1, \text{ when } 2 \le x \le 4, \\ -|x-7|+1, \text{ when } 6 \le x \le 8, \\ \quad 0, \qquad\qquad \text{otherwise.} \end{cases} \tag{4}$$

The initial value $u(x,0)$, $\quad x > 0$, is shown in the figure.

(a) Determine the solution $u(x,t)$ of the initial boundary value problem (1)-(4).

(b) Determine $\max_{t>0,x>0}(u(x,t))$. Find all points (x_*, t_*) such that

$$u(x_*, t_*) = \max_{t>0,x>0}(u(x,t)).$$

[20 marks]

2. (a) Consider the function

$$f(\mathbf{x}) = \frac{1}{2\pi}\log(|\mathbf{x}|^{-1}),\ \mathbf{x} = (x_1, x_2) \in \mathbf{R}^2.$$

Verify that

$$\Delta f(x) + \delta(x) = 0$$

in the sense of distributions. Here $\delta(\mathbf{x})$ is the Dirac delta-function, and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is Laplace's operator in two dimensions.

(b) Find a solution of the boundary value problem

$$\Delta u(\mathbf{x}) + \delta(\mathbf{x} - \mathbf{A}) = 0, \ \mathbf{x} \in \{(x_1, x_2) : x_1 > 0, x_2 > 0\},$$

$$u(x_1, 0) = 0, \quad x_1 \geq 0,$$

$$u(0, x_2) = 0, \quad x_2 \geq 0,$$

where $\mathbf{A} = (3, 1)$ (see the diagram).

[20 marks]

3. Let $u(x,t)$ be a smooth solution of the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad x \in (0,l), \ t > 0,$$

and assume that $u(x,t)$ satisfies the following boundary conditions

$$u(0,t) = 0, \ \frac{\partial u}{\partial x}(l,t) = 0, \quad \text{for all} \ \ t > 0.$$

Prove that for all possible initial condition and for any $t_2 \geq t_1 > 0$ the function $u(x,t)$ satifies the inequality

$$\int_0^l u^2(x,t_1)dx - \int_0^l u^2(x,t_2)dx \geq 0.$$

[20 marks]

4. (a) Let $f(x)$ be a bounded function such that $f(x) \to 0$ as $|x| \to \infty$. Verify that the function

$$u(x,y) = \frac{1}{\pi} \int_{-\infty} \infty \frac{y f(s) ds}{(s-x)^2 + y^2}$$

represents a solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \mathbf{x} \in \mathbf{R}, \ y > 0, \tag{5}$$

$$u(x,0) = f(x), \quad x \in \mathbf{R}. \tag{6}$$

(b) Let

$$f(x) = \begin{cases} -|x| + 2, & -2 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find a solution $u(x, y)$ of (5), (6) such that $u(x, y) \to 0$ as $x^2 + y^2 \to \infty$.

[20 marks]

5. (a) Give the definition of convolution $(f * g)$ of distributions f and g.

(b) Let $f \in \mathcal{D}'(\mathbf{R}^n)$ be a distribution. Prove that

$$\delta * f = f * \delta = f,$$

where $\delta(\mathbf{x})$ is the Dirac delta function.

(c) Let $L_N(D_x) = \sum_{|\alpha| \le N} a_\alpha D_x^\alpha$ be a linear differential operator of order N with constant coefficients a_α. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the multi-index; $\alpha_j \ge 0,\ j = 1, \ldots, n;\ |\alpha| = \alpha_1 + \ldots + \alpha_n$,

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

Let $G(\mathbf{x})$ solve the partial differential equation

$$L_N(D_x) G(\mathbf{x}) = \delta(\mathbf{x}).$$

Prove that a solution $u(\mathbf{x})$ of the equation

$$L_N(D_x) u(\mathbf{x}) = f(\mathbf{x})$$

can be represented in the form

$$u(\mathbf{x}) = (G * f)(\mathbf{x}).$$

[20 marks]

6. Consider the second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4y\frac{\partial^2 u}{\partial y^2} = 0. \tag{7}$$

In the half-planes $\{(x,y) : x \in \mathbf{R}, y > 0\}$ and $\{(x,y) : x \in \mathbf{R}, y < 0\}$, give the classification of (7), and by introducing new variables $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ reduce (7) to a canonical form.

[20 marks]

7. Find a solution $u(x,t)$ of the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad x \in (0,1), \ t > 0, \\ u(x,0) &= 2x^2, \quad x \in (0,1), \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) &= 1, \quad \text{for all } t > 0. \end{aligned} \tag{8}$$

[20 marks]