1. Five speed cameras A, C, D, E, F in the streets of a town and a base B were connected by old cables. The lengths (in hundreds yards) of each direct interconnection are shown in the table (∞ means no direct connection). The police decided to replace some cables by new ones and to remove the remaining cables. Several plans are being considered.

	A	B	C	D	E	F
A	0	∞	25	∞	26	13
B	∞	0	9	7	40	20
C	25	9	0	12	8	14
D	∞	7	12	0	∞	28
E	26	40	8	∞	0	30
F	13	20	14	28	30	0

(i) Find a minimal length layout connecting all the locations. Determine its total length. [3 marks]

Layouts containing either the shortest or second shortest path from certain cameras to the base are also considered.

(ii) Using Dijkstra's algorithm find the shortest paths from each camera to B.

[6 marks]

(iii) Identify the camera from which the layout of the cables in (i) gives neither the shortest nor second shortest connection to B. [6 marks]

(iv) Determine the lengths of the following layouts.The first is the minimal possible layout connecting all the cameras and the base and containing the shortest path to B from the camera determined in (iii). The second is the minimal possible layout connecting all the cameras and the base and containing the second shortest path to B from the camera determined in (iii). Find the difference between the lengths of these two layouts.

[5 marks]

2. The lengths of the edges (i,j) in an undirected graph with five nodes are given by the matrix:

$$\begin{pmatrix} 0 & 5 & \infty & 1 & 3 \\ 5 & 0 & 3 & 2 & \infty \\ \infty & 3 & 0 & 5 & 2 \\ 1 & 2 & 5 & 0 & 7 \\ 3 & \infty & 2 & 7 & 0 \end{pmatrix}.$$

(i) Use Floyd's method to find the lengths $l(i,j)$ of the shortest paths between all pairs i,j of nodes. Find the matrix of the predecessors.

[6 marks]

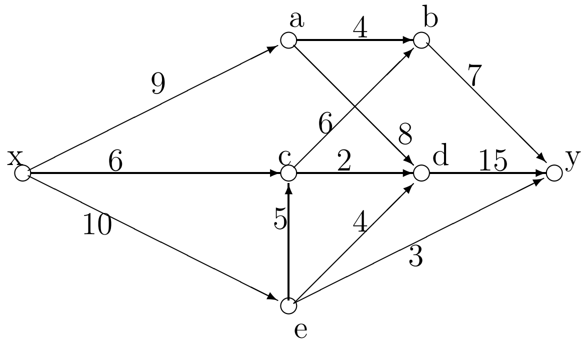
(ii) Find the length of a minimal postman tour of this graph, and find an explicit tour of this length. [6 marks]

(iii) State the necessary and sufficient conditions of the existence of an Euler tour in a directed graph. State the necessary and sufficient conditions of the existence of a postman tour in a directed graph. [4 marks]

(iv) Duplicate some edges in the graph (i) and assign directions to all edges of the amended graph so that the resulting **directed** graph admits an Euler **path**. What is the minimal possible length of edges to duplicate? Give reasons for your answer. Explain, how to asssign the directions.

[4 marks]

3. (a) In a standard network shown below, the number beside each edge is its capacity.

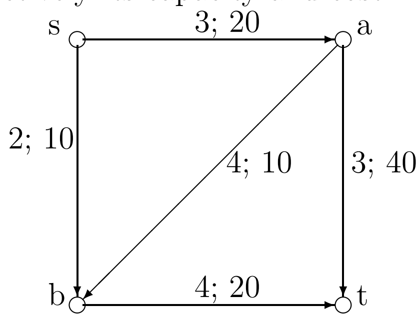


(i) Use Berge's superior path method to find a complete flow in the network.

(ii) Starting from this flow, use the Ford–Fulkerson algorithm to find a maximal flow in the network.

(iii) Find the associated cut for the maximal flow, and confirm the "minimal cut – maximal flow" theorem. [13 marks]

(b) The problem to find the maximal flow at least cost in the network shown below is stated as a minimization problem in Integer Linear Programming. What is the number of the variables? What are the types of the constraints imposed? (You do not need to write down the constrains themselves). Use the method of least cost chains and find the maximal flow at least cost. The numbers beside each edge are respectively its capacity and cost. [7 marks]



4. (a) An estate agency has invited sealed bids for four lots of land for further development and has received bids from four developers. Each developer has submitted a bid for each lot of land, on the basis that he is prepared to buy only one. The bids are shown in the matrix below: developer D_i is ready to pay p_{ij} (million pounds) for lot L_j. Use the method of lines to find an optimal assignment which bids to accept to **maximize** the total profit of all deals. [8 marks]

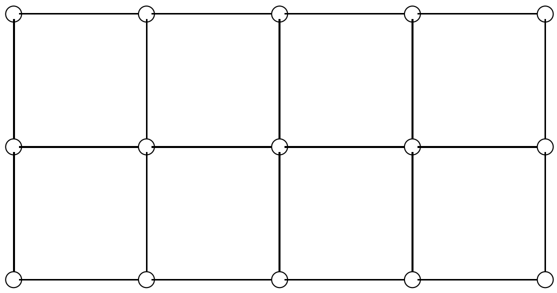
	L_1	L_2	L_3	L_4
D_1	4	6	8	3
D_2	5	4	5	6
D_3	3	3	6	4
D_4	6	7	6	8

(b) However, agency decided to sell an extra lot L_R of land for an access road. It has to be sold either together with lot L_2 or with L_4 only. Each developer submitted the extra bid for the lot L_R shown in the right column of the amended matrix. Each wants to buy either a lot of land from $L_1, \ldots, L_4$ or a lot of land for a development together with the lot for the road.

	L_1	L_2	L_3	L_4	L_R
D_1	4	6	8	3	5
D_2	5	4	5	6	8
D_3	3	3	6	4	6
D_4	6	7	6	8	6

Use the Branch and Bound method to solve this new maximization assignment problem starting with branching on those developers who are to get the new lot L_R. Show that the results of (a) can be used to get certain upper and lower bounds. Hence, find the optimal solution of this Generalized Assignment Problem. [12 marks]

5. (a) State the necessary condition for the existence of a Hamilton tour in a connected non-directed graph. Decide and give reasons whether there is a Hamilton tour in the following graph (of the shape of a grid). Give reasons for your decision.



[7 marks]

(b) Use Little's method to solve the travelling salesman's problem in the graph with the following (asymmetric) cost matrix:

$$\begin{pmatrix} \infty & 9 & 12 & 6 & 8 \\ 9 & \infty & 4 & 6 & 7 \\ 12 & 4 & \infty & 5 & 13 \\ 9 & 7 & 11 & \infty & 8 \\ 4 & 8 & 6 & 3 & \infty \end{pmatrix}.$$

[13 marks]

6. (a) Consider the Lagrange dual of the following knapsack problem:

Minimise $P = 4x_1 + 2x_2 + 5x_3 + 3x_4 + x_5$ under the constraints

$7x_1 + 3x_2 + 4x_3 + 2x_4 + 3x_5 \geq C, \qquad x_i \in \{0; 1\},\ i = 1, 2, \ldots, 5$. Constant C can take either value 13 or 14. Choose the value of C so that the solution of the knapsack problem does not require further application of the Branch and Bound procedure. Give reasons for your choice. Hence, or otherwise, solve the knapsack problem for this value of C. [10 marks]

(b) The goverment has to buy a set of medications to protect citizens agains five deseases D_1, D_2, D_3, D_4 and D_5. Medications M_1, M_2, M_3, M_4, M_5 are available. The cost of M_1 is 8 millions pounds. It protects against D_2, D_3, D_5.

The cost of M_2 is also 8 millions pounds. It protects against D_1 and D_5.

The cost of M_3 is 15 millions pounds. It protects against D_1, D_2 and D_3.

The cost of M_4 is 20 millions pounds. It protects against D_2 and D_4.

Finally, the cost of M_5 is 14 millions pounds. It protects against D_1, D_2, D_3, D_5.

State the corresponding Set Covering Problem: write down the cost vector and the cover matrix. Reduce the problem to that with a 2×3 matrix.

Find the optimal (cheapest) solution. Write down the objective for the Lagrange dual problem of the reduced Set Covering problem. Assuming that all the Lagrange multipliers are equal, find an optimal solution of the Lagrange dual problem. [10 marks]

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7. (a) Show that the bipartite graph $K_{3,3}$ is not planar. [6 marks]

(b) Each morning 3 school buses (each of capacity 24 people) start from the school S to collect pupils from six neigbouring villages. The numbers of pupils in the villages are 4, 8, 3, 12, 9 and 8 respectively.

The villages are located approximatively as follows: the first village – to the South-East of S, the second – to the South, the third – to the South-West of S, the fourth – to the West, the fifth – to the North, and the sixth – to the East of the school. The distances (in miles) between the villages and between the school and each village are given by the table:

	S	1	2	3	4	5	6
S	–	35	25	20	30	27	20
1		–	26	41	62	56	24
2			–	18	38	51	32
3				–	22	42	37
4					–	40	50
5						–	34

(i) Use phase I of the SWEEP algorithm, with the clockwise sweeping only, to find a solution to the Vehicle Routing Problem. Compare the results with the possible overall distance covered if only two buses are available.

[6 marks]

(ii) Apply the Savings Method of Clarke and Wright to the same problem. Compare the result with that of (i). Give the definition of a geographic graph.

[8 marks]