1. In winter roads connecting villages U, V, W, X, Y, Z in a mountain region are often covered with snow. The lengths of roads (in miles) are shown in the table (∞ means no direct road).

	U	V	W	X	Y	Z
U	0	18	8	13	7	14.5
V	18	0	13	20	4	∞
W	8	13	0	∞	18	∞
X	13	20	∞	0	5	3
Y	7	4	18	5	0	7
Z	14.5	∞	∞	3	7	0

Cleaning a mile of roads XZ or YV costs 200 pounds while cleaning a mile of any other road costs 100 pounds.

(i) Find a layout connecting all the villages with the minimal cost of cleaning. Determine its total cost. [4 marks]

People from a certain village complain that the minimal cost layout of (i) provides neither shortest nor second shortest path to the community center W.

(ii) Using Dijkstra's algorithm find the shortest paths from each village to W.

[6 marks]

(iii) Identify the village from which the layout in (i) gives neither the shortest nor second shortest connection to W. [6 marks]

(iv) Determine the minimal possible cost of the layout connecting all the villages and containing the shortest path to W from the village determined in (iii).

[4 marks]

2. The lengths of the edges (i, j) in an undirected graph with six nodes are given by the matrix:

$$\begin{pmatrix} 0 & 2 & 1 & \infty & 7 & 7 \\ 2 & 0 & 4 & 1 & \infty & 4 \\ 1 & 4 & 0 & \infty & \infty & 8 \\ \infty & 1 & \infty & 0 & 3 & 7 \\ 7 & \infty & \infty & 3 & 0 & 9 \\ 7 & 4 & 8 & 7 & 9 & 0 \end{pmatrix}.$$

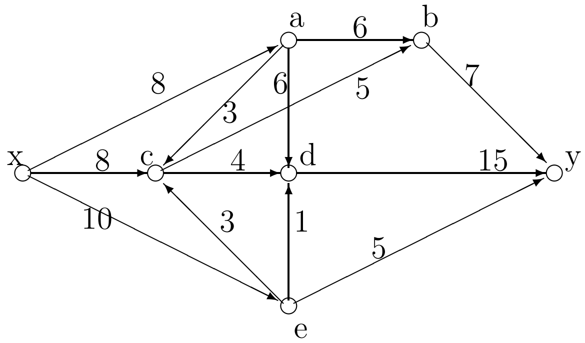
(i) Use Floyd's method to find the lengths $l(i, j)$ of the shortest paths between all pairs of nodes i, j. (You do not need to find the paths themselves).
[6 marks]

(ii) Find the length of a minimal postman tour of this graph, and describe in detail how to find a tour of this length. (You do not need to exhibit an explicit tour).
[4 marks]

(iii) State the necessary and sufficient conditions for the existence of an Euler path in a directed graph. Say briefly how they are related to the existence of an Euler tour in a modified graph.
[4 marks]

(iv) Assume we need to duplicate some edges in the graph (i) and to assign directions to all edges of the amended graph so that the resulting **directed** graph admits an Euler **path**. What is the minimal possible number of edges to duplicate? Give reasons for your answer. What is the minimal possible number of edges to duplicate if we need to have an Euler **tour** in the amended graph?
[6 marks]

3. (a) In a standard network shown below, the number beside each edge is its capacity.

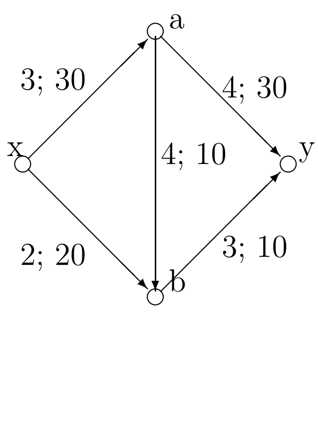


(i) Use Berge's superior path method to find a complete flow in the network.

(ii) Starting from this flow, use the Ford–Fulkerson algorithm to find a maximal flow in the network.

(iii) Find the associated cut for the maximal flow, and confirm the "minimal cut – maximal flow" theorem. [13 marks]

(b) Formulate the problem of finding the maximal flow at least cost in the network shown below (the numbers beside each edge are respectively its capacity and cost) as a minimization problem in Integer Linear Programming. Describe briefly a method you know and apply it to find the solution. [7 marks]



4. (a) The profit p_{ij} of using device D_i to perform task T_j is shown in the table below (four devices and four tasks). Use the method of lines to find an optimal assignment which **maximizes** the total profit when all the tasks are completed (one task per device). [8 marks]

	T_1	T_2	T_3	T_4
D_1	8	9	9	2
D_2	5	5	7	6
D_3	4	3	7	4
D_4	5	6	7	8

(b) To perform an additional task T_5 an extra column was added to the matrix.

	T_1	T_2	T_3	T_4	T_5
D_1	8	9	9	2	5
D_2	5	5	7	6	8
D_3	4	3	7	4	6
D_4	5	6	7	8	6

Now the restrictions become as follows: each task can be performed by a single device. Each device can perform tasks in turn one after the other. The times (in hours) required to perform tasks $T_1, \ldots, T_5$ are respectively 6,5,6,4 and 3. All tasks ought to be finished during one working day of 8 hours.

Use the Branch and Bound method to solve this new maximization assignment problem starting with branching on those devices that are assigned to the new task T_5. Use time restrictions to show that the results of (a) can be used to get certain upper and lower bounds. Hence, find the optimal solution of this Generalized Assignment Problem. [12 marks]

5. (a) Decide (and give reasons) whether there is a Hamilton tour in a connected non-directed graph whose vertices have degree 2 except for one vertex which has degree 4. [6 marks]

(b) Use Little's method to solve the travelling salesman's problem in the graph with the following (asymmetric) cost matrix:

$$\begin{pmatrix} \infty & 15 & 17 & 10 & 15 \\ 10 & \infty & 4 & 3 & 7 \\ 16 & 6 & \infty & 4 & 3 \\ 9 & 2 & 10 & \infty & 8 \\ 6 & 8 & 3 & 10 & \infty \end{pmatrix}.$$

[14 marks]

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6. (a) Solve the Lagrange dual of the following knapsack problem:

Minimise $P = 4x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5$ under the constraints

$5x_1 + 3x_2 + 8x_3 + 6x_4 + 6x_5 \geq 20, \qquad x_i \in \{0; 1\},\ i = 1, 2, \ldots, 5.$

Give reasons why, for the data given, the solution of the knapsack problem does not require further application of the Branch and Bound procedure. Hence, or otherwise, solve the knapsack problem. [11 marks]

(b) Reduce the Set Covering Problem with cost vector $(10, 10, 18, 25, 25, 20)$ and cover matrix given below to the problem with a 2×3 matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	1	0	1	0	1	1
R_2	0	0	0	0	1	0
R_3	1	1	1	0	0	0
R_4	0	1	1	0	1	1
R_5	1	0	1	1	0	1
R_6	0	1	1	1	0	1

Find the optimal (cheapest) solution. Write down the objective for the Lagrange dual problem of the reduced Set Covering problem. By restricting all the Lagrange multipliers to be equal, find an optimal solution of the Lagrange dual problem. [9 marks]

7. (a) Show that the complete graph K_5 is not planar. [6 marks]

(b) A special environment protection vehicle (of capacity 21 containers) collects daily waste from six chemical laboratories, located around the waste terminal T. The numbers of containers to collect are respectively 12, 9, 3, 8, 2 and 8. The vehicle can perform at most three runs a day.

The laboratories are located approximatively as follows: the first one – to the North of T, the second – to the East, the third – to the South-East of T, the fourth – to the South, the fifth – to the West, and the sixth laboratory – to the North-West of T. The distances (in miles) between the laboratories and between the terminal and each laboratory are given by the table:

	T	1	2	3	4	5	6
T	–	12	16	17	10	13	17
1		–	21	26	22	19	12
2			–	13	19	28	30
3				–	14	28	34
4					–	17	25
5						–	12

(i) Use phase I of the SWEEP algorithm, with clockwise sweeping only, to find a solution to the Vehicle Routing Problem. Compare the results with the possible overall distance covered if only two runs of the vehicle are available.

[7 marks]

(ii) Apply the Savings Method of Clarke and Wright to the same problem. Compare the result with that of (i).

[7 marks]