1. Five terminals A, B, C, D, E are to be connected to a server S. The lengths of cable required for each direct interconnection are shown in the table (∞ means that a direct connection is unavailable).

	S	A	B	C	D	E
S	0	18	30	∞	30	11
A	18	0	15	∞	8	9
B	30	15	0	1	11	7
C	∞	∞	1	0	∞	4
D	30	8	11	∞	0	9
E	11	9	7	4	9	0

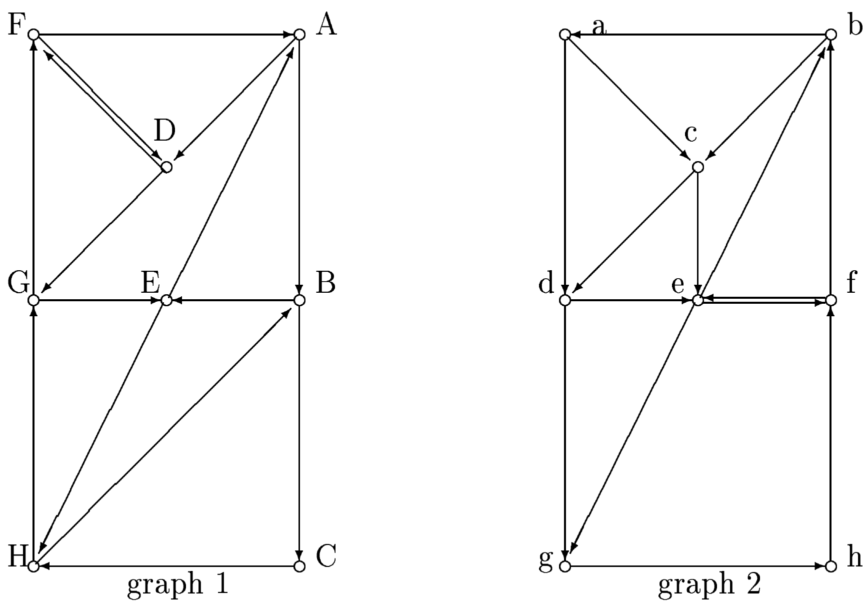
(i) Find a minimal length layout connecting all the devices. Determine its total length. [5 marks]

(ii) Using the Dijkstra method find a layout which minimizes all connections from each terminal to the server. [7 marks]

(iii) Identify terminals from which the layout in (i) does not give the shortest connection to the server. Locate those for which (i) provides the second shortest path to the server. [8 marks]

2. (a) Use graph theory to show that at any instant the number of teams in the FA Premiership each of which has played an odd number of games is even. [4 marks]

(b) For each of the directed graphs given, either find an Euler tour or show that such a tour does not exist. [6 marks]



(c) Aggregate these two graphs into one connected graph which admits an Euler tour, using some of the following extra directed edges Aa, Ab, Bb, Bd, Cd, dA, gC and gH. [5 marks]

(d) Connect the two graphs by adding only the directed edges Aa and gH. Find the smallest set of edges which must be left out of this graph so as to allow an Euler tour of the remaining edges, and give an explicit tour of the resulting graph. [5 marks]

3. The lengths of the edges (i, j) in an undirected graph with six nodes are given by the matrix:

$$\begin{pmatrix} 0 & 5 & \infty & 3 & 4 & \infty \\ 5 & 0 & 10 & 7 & \infty & 9 \\ \infty & 10 & 0 & 6 & 8 & 1 \\ 3 & 7 & 6 & 0 & \infty & \infty \\ 4 & \infty & 8 & \infty & 0 & 5 \\ \infty & 9 & 1 & \infty & 5 & 0 \end{pmatrix}.$$

(i) Use Floyd's method to find the lengths $l(i, j)$ of the shortest paths between all pairs of nodes i, j. (You do not need to find the paths themselves).

[7 marks]

(ii) Find the length of a minimal postman tour of this graph, and say how to find a tour of this length. (You do not need to exhibit an explicit tour).

[8 marks]

(iii) What are the conditions which imply the existence of a postman tour in a *directed* graph? Give an example of a directed graph in which one of the edges must be visited at least 5 times by any postman tour.

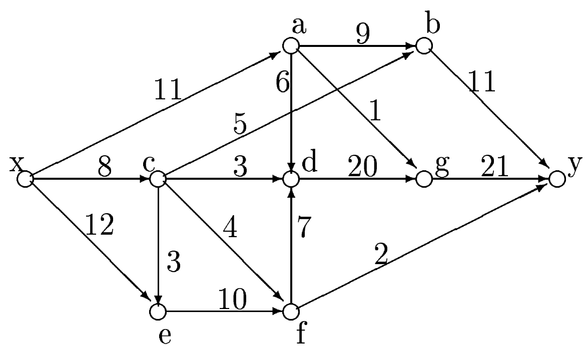
[5 marks]

4. (a) In a standard network N, with source x and sink y, an edge (u, v) has capacity $c(u, v)$.

Define the capacity of a cut. Define an overall value $f(N)$ of a flow f. Show that $f(N)$ does not exceed the capacity of any cut.

[7 marks]

(b) In the standard network shown below, the number beside each edge is its capacity.



(i) Use Berge's superior path method to find a complete flow in the network.

(ii) Starting from this flow, use the Ford – Fulkerson algorithm to find a maximal flow in the network.

(iii) Find the associated cut for the maximal flow, and confirm the "minimal cut – maximal flow" theorem.

[13 marks]

5. A collection of n devices are to be used for r tasks $T_1, \ldots, T_r$. Each device M_i can be used for a subset S_i of the tasks.

(a) Describe in detail how the maximum flow techniques in a standard network can be used to determine whether or not it is possible to assign devices to all tasks so that each device would be used exactly in one task for which it is capable.

Consider the cases:

(i) $r = n$;

(ii) each task may be carried out using at most three devices, while each device may work on exactly one task.

[7 marks]

(b) The cost t_{ij} of device M_i to perform task T_j ($n = r = 4$) is shown in the table below.

Describe the method and find an optimal assignment which minimizes the total cost to complete all the tasks (one task per each device).

	T_1	T_2	T_3	T_4
M_1	23	17	17	18
M_2	22	18	16	26
M_3	19	24	14	22
M_4	21	24	23	20

[13 marks]

6. (a) Formulate the necessary condition for the existence of a Hamilton tour in an undirected graph.

[5 marks]

(b) Use Little's method to solve the travelling salesman's problem in the graph with the following (asymmetric) cost matrix:

$$\begin{pmatrix} \infty & 10 & 16 & 19 & 12 \\ 5 & \infty & 8 & 36 & 25 \\ 23 & 19 & \infty & 21 & 5 \\ 7 & 30 & 24 & \infty & 6 \\ 18 & 11 & 10 & 21 & \infty \end{pmatrix}.$$

[15 marks]

7. (a) Show that the complete graph K_5 is not planar.

[6 marks]

(b) Six petrol stations are supplied from a refinery R. Their requirements are 8, 8, 3, 5, 4 and 2 tonnes of petrol per day respectively. Three vehicles are available, each of capacity $Q = 15$ tonnes. The stations are located approximatively as follows: the first station – to the North of R, the second – to the East, the third – to the South-East of R , the fourth – to the South, the fifth – to the West, and the sixth – to the North-West of R. The distances (in miles) between the stations and between the refinery R and each station are given by the table:

	R	1	2	3	4	5	6
R	–	12	16	12	10	13	7
1		–	21	23	22	19	10
2			–	15	19	28	22
3				–	10	23	19
4					–	17	17
5						–	8

(i) Use phase I of the SWEEP algorithm, with clockwise sweeping only, to find a solution to the Vehicle Routing Problem.

[7 marks]

(ii) Use the Savings Method of Clarke and Wright to the same problem. Compare the result with that of (i).

[7 marks]