1. A plumber installing new water pipes to connect the hot water storage tank S in a house to bedrooms A, B, C, D, E prepared an estimate of the lengths of pipe, in metres, required for each direct interconnection. These lengths are shown in the table below, where the entry ∞ means that a direct connection was not thought practicable.

	S	A	B	C	D	E
S	0	∞	10	∞	18	∞
A	∞	0	10	16	9	2
B	10	10	0	9	8.5	5
C	∞	16	9	0	7	∞
D	18	9	8.5	7	0	11
E	∞	2	5	∞	11	0

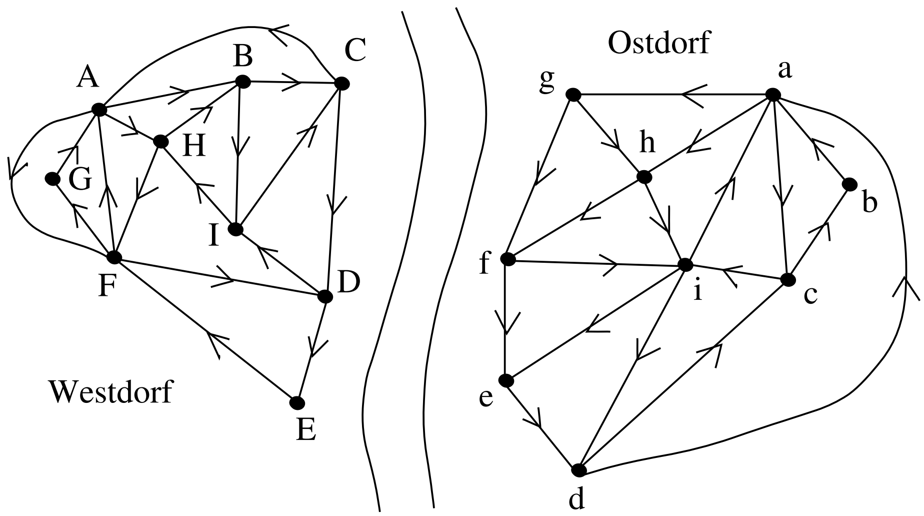
(i) Assuming that the plumber charges a fixed price per metre, find the least cost layout needed to supply all the rooms with hot water, and the cost of this layout. [5 marks]

(ii) The plumber suggests that it would be better if the pipe run to each individual room were as short as possible. Use Dijkstra's algorithm to determine the layout under these conditions, and calculate the extra length of piping needed, compared with the least cost layout. [7 marks]

(iii) Identify those rooms for which the least cost layout in (i) does not give the shortest supply route, and for each room say, with reasons, whether (i) gives the second shortest route. [8 marks]

2. The picturesque old town of Hausdorf is divided by a river into two parts, Westdorf and Ostdorf. Streets in the historic parts are narrow and allow one-way traffic only.

The graph below shows the streets and junctions in the touristic parts of each half of the town.



A coach operator arranges coach tours of Westdorf starting and ending at A, while another coach operator provides tours of Ostdorf starting and ending at a. Since all streets contain sights of historic interest the tours are arranged to travel along every street in their half of the town.

(i) Exhibit a route for the tour of Westdorf which passes down each street once only, if one exists, and otherwise explain why there is no such route. [4 marks]

(ii) Do the same for the tour of Ostdorf. [3 marks]

On special occasions the operators offer a grand tour of both Westdorf and Ostdorf, using some of the many bridges across the river. The bridges can be represented by edges Ca, Cg, Df, Ee, dE, eD and fC. Each bridge ij allows one-way traffic only, in the direction from i to j.

(iii) Determine whether there is a route for the grand tour starting and ending at A which visits all streets in Westdorf and Ostdorf once only. The tour may cross a bridge more than once or not at all. [5 marks]

(iv) Show that it is impossible to find a grand tour which crosses *all* the bridges at least once, and visits all the streets once and once only. [3 marks]

(v) Show that there is a tour which crosses all the bridges at least once and visits all streets once only except for one, and identify the street to be omitted. [5 marks]

3. (a) In an undirected graph G with six nodes the length d_{ij} of the edge between nodes i and j is given by the symmetric matrix

$$D = (d_{ij}) = \begin{pmatrix} 0 & 2 & 7 & 8 & \infty & \infty \\ 2 & 0 & 3 & 9 & 10 & \infty \\ 7 & 3 & 0 & 4 & \infty & \infty \\ 8 & 9 & 4 & 0 & 5 & 2 \\ \infty & 10 & \infty & 5 & 0 & 4 \\ \infty & \infty & \infty & 2 & 4 & 0 \end{pmatrix}.$$

(i) Use Floyd's method to find the length l_{ij} of the shortest path between i and j, for all pairs of nodes. [7 marks]

(ii) Find the length of a minimal postman's tour of G, and say how to find a tour of this length. (You do not need to exhibit an explicit tour). [8 marks]

(b) Show that in any minimal postman's tour of a connected undirected graph each edge is visited at most twice.

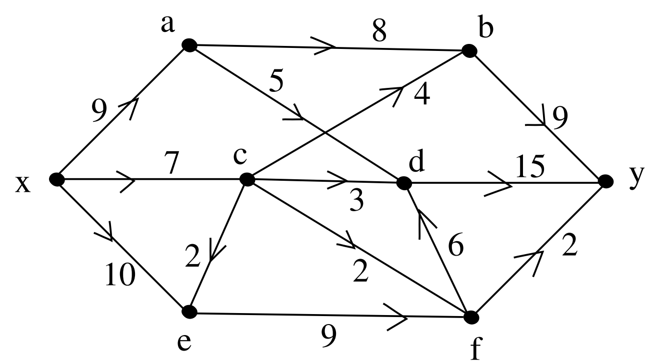
Give an example of a *directed* graph in which one edge must be visited three times by any postman's tour. [5 marks]

4. (a) Let N be a standard network, with source x and sink y, in which the edge (u, v) has capacity $c(u, v)$.

(i) Write down the conditions which a flow f, with value $f(u, v)$ on the edge (u, v), has to satisfy, and write down the value $f(N)$ of the flow through the network.

(ii) Define the partition $(P_f, \overline{P}_f)$ associated with f, and show that if $f(N)$ is maximal then the associated partition is a cut. [7 marks]

(b) In the standard network shown below, the number beside each edge is its capacity.

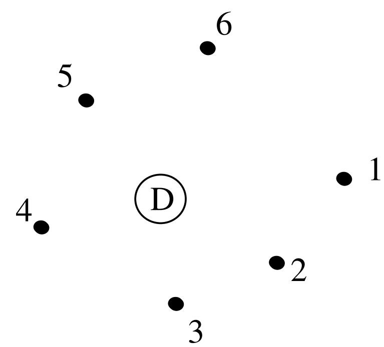


(i) Use Berge's superior path method to find a complete flow in N.

(ii) Starting from this flow, use the Ford-Fulkerson algorithm to find a maximal flow in N.

(iii) Find the associated cut for the maximal flow, and confirm that the capacity of the cut equals the value of the maximal flow. [13 marks]

5. Six customers are arranged round a depot approximately as shown.



Distances, in miles, between customers, and between the depot and each customer are given as follows:

	D	**1**	**2**	**3**	**4**	**5**	**6**
D	–	12	6	12	18	12	10
1		–	8	19	24	23	14
2			–	11	16	18	15
3				–	18	26	24
4					–	16	22
5						–	10

Customer requirements, in tonnes, are given by $\mathbf{q} = (4, 2, 8, 8, 3, 5)$, and 3 vehicles are available, each of capacity $Q = 15$ tonnes.

(i) Use phase I of the SWEEP algorithm, with clockwise sweeping only, to find a solution to the Vehicle Routing Problem, with the data given. [7 marks]

(ii) Many of the possible solutions found in (i) involve approximately the same distance. From among these select a solution with the smallest value for the *range* between the longest and the shortest distance travelled by the three vehicles. [2 marks]

(iii) Determine whether there is any two-vehicle solution to the routing problem; if so, give this solution, and compare the total distance required to your best three-vehicle solution found above. [4 marks]

(iv) Use the Savings Method of Clarke and Wright to find a solution to the Vehicle Routing Problem with the data given in (i), and comment on whether you should expect the same solution as found in (i). [7 marks]

6. (i) Give two applications of the travelling salesman's problem in which the weight function may fail to be symmetric. [4 marks]

(ii) Use Little's method to solve the asymmetric travelling salesman's problem when the cost matrix is

$$C = \begin{pmatrix} \infty & 11 & 14 & 20 & 13 \\ 6 & \infty & 10 & 37 & 26 \\ 24 & 20 & \infty & 17 & 6 \\ 5 & 48 & 22 & \infty & 4 \\ 19 & 12 & 11 & 22 & \infty \end{pmatrix}$$

[16 marks]

7. A team of n people, $M_1, \ldots, M_n$ are to work on a project involving r tasks, $T_1, \ldots, T_r$. Each team member M_i is trained to carry out a subset S_i of the tasks.

(a) Describe in detail how you could use maximum flow techniques in a standard network to determine whether or not it is possible to assign tasks to team members so that each team member carries out exactly one task for which they are trained, in the case $r = n$.

Explain how to modify the approach to deal with the case $r \neq n$, where

(i) each member must work on exactly one task, while each task may be carried out by at most two people working together,

(ii) each task is carried out by exactly one person, while each person may work on up to three tasks. [8 marks]

(b) An estimate of the time t_{ij} needed for team member M_i to perform task T_j has been made, (setting $t_{ij} = \infty$ when M_i cannot do the task). An optimal assignment of tasks in the case $r = n$ is an assignment of exactly one task to each person that minimises the total estimated time to complete all the tasks.

Find an optimal assignment when the time estimates are given in the table below, giving full details of the method used.

[A choice of assignment, without reasons, will get very little credit]

	T_1	T_2	T_3	T_4
M_1	14	17	19	18
M_2	22	18	24	26
M_3	17	16	23	22
M_4	21	24	23	23

[12 marks]