Full marks may be obtained for complete answers to **four** questions.

Credit will only be given for the best **four** answers.

**Useful formulae**

1) For any two random variables, $U$ and $W$, the covariance between $U$ and $W$, cov($U$, $W$), is defined by

$$\text{cov}(U, W) = E(UW) - E(U)\,E(W)$$

and correlation, $\text{corr}(U, W)$, between $U$ and $W$ is defined by

$$\text{corr}(U, W) = \text{cov}(U, W) / \{V(U)V(W)\}^{\frac{1}{2}}$$

2) If $\{x_t\}(t = 0, \pm 1, \ldots,)$ is a weakly stationary process with covariance function $R(u)\,(t, u = 0, \pm 1, \ldots)$ its spectral density function, $f(\lambda)$, is defined by

$$f(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} R(u) \exp(-i\,u\,\lambda)$$

for all values of $\lambda$ for which the sum on the right hand exists.

1(a) An information signal may be represented by the sequence

$$x_t = A\sin(\omega t + \Theta) \qquad (t = 0, \pm 1, \dots),$$

where ω and $A$ are fixed constants and $\Theta$ is a uniformly distributed random variable with probability density function

$$p_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi \le \theta \le \pi, \\ 0 & \text{elsewhere.} \end{cases}$$

Show that $\{x_t\}$ is weakly stationary with covariance function

$$\operatorname{cov}(x_t, x_{t+u}) = \tfrac{1}{2} A^2 \cos(\omega u) \quad (t, u = 0, \pm 1, \dots).$$ [12 marks]

(b) It has been suggested, however, that due to transmission noise, the amplitude, $A$, may not be treated as fixed and the signal actually being transmitted is $y_t$, where

$$y_t = A \sin(\omega t + \Theta) \qquad (t = 0, \pm 1, \dots)$$

and A is a random variable, independent of Θ, following a Rayleigh distribution with probability density function

$$p_A(a) = \begin{cases} a \exp\left(-\tfrac{1}{2} a^2\right) & a > 0, \\ 0 & a \le 0. \end{cases}$$

Show that $\{y_t\}$ is still stationary but with covariance function

$$\operatorname{cov}(y_t, y_{t+u}) = \cos(\omega u) \qquad (t, u = 0, \pm 1, \dots).$$ [5 marks]

[N.B. You may assume without proof the results that $E(A) = \sqrt{\tfrac{\pi}{2}}, \quad E(A^2) = 2.$]

(c) Suppose instead that the information signal being transmitted is $z_t$, where

$$z_t = C_t \sin(\omega t + \Theta) \qquad (t = 0, \pm 1, \dots),$$

$\{C_t\}$, independent of Θ, is a stationary process with mean 0 and an absolutely summable covariance function $R_C(u)$. Show that $\{z_t\}$ is also stationary with covariance function

$$R_z(u) = \operatorname{cov}(z_t, z_{t+u}) = \tfrac{1}{2} R_C(u) \cos(\omega u) \qquad (t, u = 0, \pm 1, \dots).$$ [3 marks]

(d) Show also that the spectral density function, $f_z(\lambda)$, of $\{z_t\}$ is related to that of $\{C_t\}$ by

$$f_z(\lambda) = \tfrac{1}{4}\{f_C(\lambda - \omega) + f_C(\lambda + \omega)\},$$

***Question 1 continued overleaf***

***Q1 continued***

where $f_C(\lambda)$ denotes the spectral density function of $\{C_t\}$. [4 marks]

Deduce that if $\omega$ is close to 0, $f_z(\lambda)$ will be close to $\frac{1}{2} f_C(\lambda)$. [1 mark]

[N.B. sin(A + B) = sinA cosB + cosA sinB,
cos(A + B) = cosA cosB - sinA sinB.]

2(a) Suppose that $\{x_t\}$ is a discrete-time (weakly) stationary process with mean $\mu_x$ and covariance function $R_x(u)$ and let $\{y_t\}$, independent of $\{x_t\}$, be a different stationary process with mean $\mu_y$ and covariance function $R_y(u)$. Let

$$w_t = x_t + y_t \qquad (t = 0, \pm 1, \dots)$$

be a new process obtained by aggregating $x_t$ and $y_t$ for each $t$. Show that $\{w_t\}$ is also stationary with covariance function $R_w(u)$, where

$$R_w(u) = R_x(u) + R_y(u) \qquad (u = 0, \pm 1, \dots).$$ [5 marks]

(b) Let $\{x_t\}(t = 0, \pm 1, \dots)$ be an autoregressive process of order 1

$$x_t = \alpha x_{t-1} + \varepsilon_t, \qquad |\alpha| < 1,$$

where $\{\varepsilon_t\}$ is a purely random sequence of uncorrelated random variables, each with mean 0 and variance $\sigma^2$. Show that $\{x_t\}$ is weakly stationary with covariance function

$$R_x(u) = \alpha^{|u|} R_x(0) \qquad (t, u = 0, \pm 1, \dots),$$

where $R_x(0) = \sigma^2 / (1 - \alpha^2)$ denotes the variance of $\{x_t\}$. [8 marks]

(c) Suppose that the weekly sales of a certain product manufactured by a major company for a given area, called Region 1, could be described, after a suitable transformation, by an autoregressive process of order 1, $\{x_t\}$, but those in a neighbouring area, called Region 2, by an independent autoregressive process of order 1, $\{y_t\}$, where

$$y_t = \beta y_{t-1} + \delta_t, \qquad |\beta| < 1,$$

and $\{\delta_t\}$, independent of $\{\varepsilon_t\}$, is also a purely random sequence of uncorrelated random variables, each with mean 0 and variance $\tau^2$. Let

$$w_t = x_t + y_t \qquad (t = 0, \pm 1, \dots)$$

be the process obtained by aggregating $\{x_t\}$ and $\{y_t\}$.

Write down the covariance function, $R_w(u)$ of $\{w_t\}$. [2 marks]

(d) Show that for all $u \geq 2$, $R_w(u)$ satisfies the equation

$$R_w(u) - (\alpha + \beta) R_w(u-1) + \alpha\beta R_w(u-2) = 0 \qquad (u = 2, 3, \dots).$$ [6 marks]

Show also that $R_w(1)$ does not satisfy this last equation. [2 marks]

Provide an interpretation of this last result for the behaviour of $\{w_t\}$. [2 marks]

3(a) Suppose that $\{x_t\}$ ($t$ = 0, ± 1, …) is a stationary process with mean 0, an absolutlety summable covariance function, $R_x(u)$, and the spectral density function $f_x(\lambda)$. Let

$$y_t = \sum_{j=-q}^{q} c_j x_{t-j} \qquad (t = 0, \pm 1, \ldots)$$

be a 'filtered' series obtained from $x_t$,; here $q \geq 0$ is an integer and the $c_j$ are some real constant whose values do not depend on $t$. Show that $\{y_t\}$ is also stationary with covariance function

$$R_y(u) = \sum_{j=-q}^{q} \sum_{k=-q}^{q} c_j \; c_k \; R_x(u + k - j),$$

and spectral density function

$$f_y(\lambda) = |C(\lambda)|^2 \; f_x(\lambda),$$

where

$$C(\lambda) = \sum_{j=-q}^{q} c_j \; e^{-ij\lambda}$$

denotes the transfer function of the filter and $|C(\lambda)|$ denotes the Gain function of the filter.

[6 marks]

(b) Consider the following two filters:

1) $y_t = \frac{1}{4} x_t + \frac{1}{2} x_{t-1} + \frac{1}{4} x_{t-2}$;

2) $y_t = \frac{1}{3} \sum_{j=-1}^{1} x_{t-j}$.

Find the transfer functions, $C_1(\lambda)$, and $C_2(\lambda)$, say, of these two filters and show that their gain functions, $G_1(\lambda)$ and $G_2(\lambda)$, are given by

$$G_1(\lambda) = |C_1(\lambda)| = \frac{1}{2}(1 + \cos\lambda),$$

$$G_2(\lambda) = |C_2(\lambda)| = \frac{1}{3}\left|\frac{\sin\frac{3}{2}\lambda}{\sin\frac{1}{2}\lambda}\right|.$$

[13 marks]

Sketch $G_1(\lambda)$ and $G_2(\lambda)$ for $\lambda \in [0, \pi]$ and describe effects these two filters will have on the behaviour of the output series $y_t$.

[6 marks]

4. A wide variety of financial time series may be approximated, to second order, that is, in terms of the behaviour of their correlation structure, by a Random Walk model, which postulates that the observed process, $\{x_t\}(t = 1, 2, ...)$ follows the model

$$x_t = x_{t-1} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a sequence of uncorrelated random variables, each with mean 0 and variance $\sigma^2$, and $x_0$ may be treated as a fixed constant.

Demonstrate that, for each $t \geq 1$, $x_t$ may be written as

$$x_t = \sum_{j=0}^{t-1} \varepsilon_{t-j} + x_0. \qquad \text{[2 marks]}$$

Hence show that for each fixed $t$ and all $k = 0, 1, \ldots, t$, the covariance function of $\{x_t\}$ is given by

$$\text{cov}\left(x_t, x_{t-k}\right) = (t-k)\,\sigma^2 \qquad (k = 0, 1, \ldots, t,\ \ t = 1, 2, \ldots). \qquad \text{[5 marks]}$$

Write down the correlation function of $\{x_t\}$ and deduce that if $k$ is small relative to $t$

$$\text{corr}(x_t, x_{t-k}) \approx 1, \qquad k > 0. \qquad \text{[3 marks]}$$

Let $\Delta x_t = x_t - x_{t-1}$ denote the first differences of $\{x_t\}$ $(t > 1)$ and

$$y_t = \Delta x_t - \Delta x_{t-1} = x_t - 2x_{t-1} + x_{t-2} \qquad (t = 2, 3, \ldots)$$

denote the second differences of $\{x_t\}$. Show that $\{y_t\}$ follows a moving average process of order 1

$$y_t = \varepsilon_t - \varepsilon_{t-1}. \qquad \text{[1 mark]}$$

Find the covariance function of $\{y_t\}$ and show that its correlation function is given by

$$\text{corr}(y_t, y_{t-k}) = \begin{cases} 1 & k = 0, \\ -\tfrac{1}{2} & k = \pm 1, \\ 0 & |k| > 1. \end{cases} \qquad \text{[6 marks]}$$

Explain why $\{y_t\}$ is stationary. Is it invertible? [2 marks]

Let

$$R_y^{(T)}(u) = \frac{1}{T} \sum_{t=1}^{T-u} y_t\, y_{t+u} \qquad (u \geq 0)$$

denote the standard 'positive definite' estimator of the covariance function, $R_y(u)$ ,of $\{y_t\}$, based on an observed realisation, $y_1, \ldots, y_T$, $T > 1$, of $\{y_t\}$.

***Question 4 continued overleaf***

***Q4 continued***

Demonstrate that

$$E\left\{R_y^{(T)}(0)\right\} = 2\sigma^2, \qquad E\left\{R_y^{(T)}(1)\right\} = -\left(\frac{T-1}{T}\right)\sigma^2,$$

and deduce that whereas $R_y^{(T)}(0)$ provides an unbiased estimator of $R_y(0)$, $R_y^{(T)}(1)$ provides a biased estimator. Find the bias, $B\left\{R_y^{(T)}(1)\right\} = E\left\{R_y^{(T)}(1)\right\} - R_y(1)$, of $R_y^{(T)}(1)$ in estimating $R_y(1)$ and show that $R_y^{(T)}(1)$ nevertheless provides an asymptotically unbiased estimator of $R_y(1)$.

[6 marks]

5. Let $\{x_t\}$ ($t = 0, \pm 1, \ldots$) be an autoregressive process of order 2:

$$x_t = x_{t-1} + \alpha x_{t-2} + \varepsilon_t, \qquad -1 < \alpha < 0,$$

where $\{\varepsilon_t\}$ is a sequence of uncorrelated random variables, each with mean 0 and variance $\sigma^2$. The process admits an infinite moving average representation

$$x_t = \sum_{j=0}^{\infty} b(j)\varepsilon_{t-j}, \qquad b(0) = 1,$$

where $b(j)$ is the coefficient of $z^j$ in the expansion of $(1 - z - \alpha z^2)^{-1}$ in non-negative powers of z, that is,

$$\sum_{j=0}^{\infty} b(j)z^j = \left(1 - z - \alpha z^2\right)^{-1}.$$

(a) Show that the correlation function, $r(u)$, of $\{x_t\}$ satisfies the Yule-Walker equations

$$r(1) = (1-\alpha)^{-1} \qquad (1)$$
$$r(2) = r(1) + \alpha, \qquad (2)$$
$$r(u) = r(u-1) + \alpha r(u-2) \quad (u \geq 3). \qquad (3)$$

[9 marks]

(b) Deduce that $r(1)$ satisfies the inequality

$$0.5 < r(1) < 1.$$

[3 marks]

Suppose that only $x_1, \ldots x_T$, $T > 1$, have been observed and $\alpha$ is unknown. Let

$$\hat{r}(u) = \sum_{t=1}^{T-u} x_t\, x_{t+u} \Big/ \sum_{t=1}^{T} x_t^{\,2} \qquad (u = 1, 2, \ldots)$$

denote a 'positive-definite' estimator of $r(u)$.

(c) Two different methods of estimating $\alpha$ from $\hat{r}(1)$ and $\hat{r}(2)$ are under consideration:

(1) Estimate $\alpha$ by $\tilde{\alpha}$, where $\tilde{\alpha}$ is based only on $\hat{r}(1)$ and it is obtained by using the relation (1) above, that is, by solving the equation

$$\hat{r}(1) = (1 - \tilde{\alpha})^{-1}.$$

(2) Estimate $\alpha$ by $\hat{\alpha}$, where $\hat{\alpha}$ is based on both $\hat{r}(1)$ and $\hat{r}(2)$ and it is obtained by using the relation (2) above, that is, by

$$\hat{\alpha} = \hat{r}(2) - \hat{r}(1).$$

Show that

$$\tilde{\alpha} = \{\hat{r}(1) - 1\} / \hat{r}(1).$$

[2 marks]

***Question 5 continued overleaf***

***Q 5 continued***

By taking some trial values of $\hat{r}(1)$, or otherwise, deduce that if $\hat{r}(1) \leq 0.5$, $\tilde{\alpha}$ does not lie in the range of value of $\alpha$ for which the process is stationary. [2 marks]

On the assumption that $x_1$ and $x_2$ may be treated as fixed, find the least-squares estimator, $\hat{\alpha}^*$, say, of $\alpha$ based on $x_3, \ldots, x_T$. [5 marks]

Explain why, for large values of $T$, the difference between $\hat{\alpha}$ and $\hat{\alpha}^*$ may be expected to be small. [4 marks]

6(a) Explain what is meant by stylised features of a financial time series. Briefly describe three such features as applicable to changes in share prices. [5 marks]

(b) Consider the model

$$x_t = \varepsilon_t + \beta\varepsilon_{t-1}\,\varepsilon_{t-2}, \qquad t = 0, \pm 1, \ldots,$$

where $\{\varepsilon_t\}$ is a sequence of independent identically distributed random variables, each with mean 0, variance $\sigma^2$. Show that $\{x_t\}$ is a purely random process with variance $\sigma^2\left(1+\sigma^2\beta^2\right)$, that is,

$$\operatorname{cov}\left(x_t, x_{t-s}\right) = \begin{cases} \sigma^2\left(1+\sigma^2\beta^2\right), & s = 0, \text{ all } t, \\ 0 & s > 0, \text{ all } t. \end{cases}$$ [6 marks]

Are $x_t$ and $x_{t-1}$ mutually independent? Explain giving reasons. [2 marks]

Comment on the relevance of this model for financial data. [2 marks]

Consider the Autoregressive Conditional Heteroscedastic model of order 1, ARCH(1) model,

$$x_t = \sigma_t\,\varepsilon_t,$$

where $\{\varepsilon_t\}$ is a sequence of independent random variables, each with mean 0 and variance 1, and the variance, $\sigma_t^2$, of $x_t$ depends upon $x_{t-1}^2$, the square of the immediate past observation, as follows:

$$\sigma_t^2 = \gamma + \alpha x_{t-1}^2 \qquad \gamma > 0,\ 0 < \alpha < 1.$$

On writing

$$x_t^2 = \sigma_t^2 + x_t^2 - \sigma_t^2,$$

show that, in this model, $x_t^2$ is postulated to follow an autoregressive model of order 1

$$x_t^2 = \gamma + \alpha\, x_{t-1}^2 + v_t,$$

where $v_t = \sigma_t^2\left(\varepsilon_t^2 - 1\right)$. [2 marks]

Explain why

$$E\left(v_t\right) = 0, \text{ all } t$$ [2 marks]

Show that

$$E\left(x_t\right) = 0, \text{ all } t;$$

$$V\left(x_t\right) = E\left(x_t^2\right) = \frac{\gamma}{1-\alpha}, \text{ all } t.$$ [4 marks]

Comment on why ARCH models may be useful for financial data. [2 marks]

7. A linear time series model with non-Normal innovations has been suggested as a suitable model for financial data. For investigating this hypothesis, suppose that the observed time series is a (part) realization of a discrete-time moving average process of order1:

$$x_t = \varepsilon_t + \beta\, \varepsilon_{t-1}, \qquad |\beta| < 1,$$

in which $\{\varepsilon_t\}$ is a sequence of independent identically distributed, not necessarily Normally distributed, random variables, each with mean 0 variance 1 and finite fourth moment $\lambda = E\left(\varepsilon_t^{\,4}\right)$.

Let

$$y_t = x_t^2.$$

Show that for each fixed *t*, where *t* is an integer,

a) $E(y_t) = 1 + \beta^2;$ [2 marks]

b) $E\left(y_t^{\,2}\right) = \left(1 + \beta^4\right)\lambda + 6\beta^2;$ [2 marks]

c) $E(y_t y_{t-1}) = \lambda\beta^2 + 1 + \beta^2 + \beta^4;$ [2 marks]

d) $E(y_t y_{t-k}) = \left(1 + \beta^2\right)^2,$ all $k \geq 2$. [2 marks]

Hence deduce that

e) $V(y_t) = (\lambda - 3)\left(1 + \beta^4\right) + 2\left(1 + \beta^2\right)^2;$ [2 marks]

f) $\operatorname{cov}(y_t, y_{t-1}) = (\lambda - 3)\,\beta^2 + 2\beta^2;$ [2 marks]

g) $\operatorname{cov}(y_t, y_{t-k}) = 0,$ all $k \geq 2$. [1 mark]

Write down the correlation function, $\operatorname{corr}(y_t, y_{t-k}),$ for all $k \geq 1$. [1 mark]

Now let

$$\kappa = \frac{E\left(x_t^{\,4}\right)}{\left\{E\left(x_t^{\,2}\right)\right\}^2}$$

denote the coefficient of Kurtosis of $\{x_t\}$.

Deduce that

h) $\kappa = \dfrac{\left(1 + \beta^4\right)\lambda + 6\beta^2}{\left(1 + \beta^2\right)^2};$ [1 mark]

***Question 7 continued overleaf***

***Q 7 continued***

i) *(κ - 3)* is related to *(λ - 3)* as follows:

$$(\kappa - 3) = \frac{(\lambda - 3)(1+\beta^4)}{(1+\beta^2)^2}$$ [2 marks]

Hence prove that

$$\frac{(\kappa - 3)\theta + 2\rho^2}{\kappa - 3 + 2} = \mathrm{corr}(y_t, y_{t-1}),$$

where $\rho = \beta / (1+\beta^2) = \mathrm{corr}(x_t, x_{t-1})$ and $\theta = \beta^2 / (1+\beta^4)$. [4 marks]

Show that if $\kappa = 3$, implying the common distribution of $\{\varepsilon_t\}$ is Normal,

$$\mathrm{corr}(y_t, y_{t-1}) = \rho^2,$$ [2 marks]

Comment on the behaviour of $\mathrm{corr}(y_t, y_{t-1})$ when $\kappa > 3$ and explain the relevance of this result for financial data. [2 marks]