1. (a) State and prove the Markov inequality. [6 marks]

   (b) State the weak law of large number (WLLN) and use the Markov inequality to prove it. [6 marks]

   (c) State the ordinary central limit theorem.

   Suppose X_n has a binomial distribution with parameters n and p. Explain why X_n can be written as a sum of independent and identically distributed random variables and derive the distribution of X_n for large n. [8 marks]

2. Let X and Y be two random variables.

   (a) Under appropriate conditions (to be stated by you), prove that

$$\text{Variance}\{X\} = \text{Variance}\{E[X|Y]\} + E\{\text{Variance}[X|Y]\}.$$

   You may use without proof the result that $E\{E[X|Y]\} = E(X)$. [6 marks]

   (b) State and prove the Rao-Blackwell theorem. Explain how can it be used to reduce the variance of an unbiased estimator based on a sufficient statistic. [8 marks]

   (c) The manufacturer of a certain type of car tyre guarantees to pay compensation if the tyre needs to be replaced within 22,000 km of its use. Assume that the distance, X, travelled before a tyre is replaced has an exponential distribution with the following probability density function (pdf)

$$f(x, \lambda) = \lambda e^{-\lambda x} I(x > 0),$$

   where λ is the unknown parameter. Suppose we are interested in estimating the probability $P[X < 22000]$ based on a random sample $X_1, \ldots, X_n$ of size n where X_i denotes the distance (in km) travelled by the i-th car in the sample, $1 \leq i \leq n$.

   Obtain an unbiased estimator of $P[X < 22000]$ based on X_1 only.

   [Hint: Use indicator function.]

   Assume that $\bar{X} = \sum_{i=1}^{n} X_i/n$ is a sufficient statistic for λ, use the Rao-Blackwell theorem to obtain an improved estimator of $P[X < 22000]$. You do not need to show the explicit expression of the estimator in terms of $\bar{X}$. [6 marks]

3. Let θ be the proportion of the population who support a certain political issue, $0 < \theta < 1$. Let $X_1, \ldots, X_n$ denote a random sample of size n from the population where $X_i = 1$ if the i-th person supports the issue and $X_i = 0$ if the i-th person does not support the issue. The probability mass function (pmf) of each X_i is given by

$$f(x, \theta) = \theta^x (1-\theta)^{(1-x)} I(x = 0, 1).$$

Suppose we are interested in estimating the proportion θ based on $X_1, \ldots, X_n$.

(a) Prove directly using the definition of sufficiency that $\sum_{i=1}^n X_i$ is a sufficient statistic for θ. Here you are not allowed to use the factorisation theorem. [8 marks]

(b) Write down the expressions for the mean and variance of the sample mean $\bar{X}$. [2 marks]

(c) Prove directly using the definition of completeness that $\sum_{i=1}^n X_i$ is a complete sufficient statistic for θ. Next use the Lehmann and Scheffe technique to derive the uniformly minimum variance unbiased estimator (UMVUE) of θ. [10 marks]

4. Let $L(\theta, x_1, \ldots, x_n)$ denote the likelihood function of a random sample of n observations $\{x_1, \ldots, x_n\}$ from a distribution $f(x, \theta)$ and let T denote an unbiased estimator of $\tau(\theta)$, a function of θ, and put $U = \frac{\partial \log L}{\partial \theta}$.

   (a) Prove that under appropriate conditions Covariance$(T, U) = \tau'(\theta) = \frac{\partial \log L}{\partial \theta}$. Here you may assume without proof that $E(U) = 0$ and $E(U^2) = -E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]$. [5 marks]

   (b) Hence derive the Cramer-Rao lower bound for the variance of an unbiased estimator based on a random sample of size n. [3 marks]

   (c) The total number of misprints per page in a book can often be modelled as a Poisson random variable X with parameter λ and the probability mass function (pmf) given by

$$f(x, \lambda) = e^{-\lambda} \lambda^x / x!, \quad x = 0, 1, 2, \ldots.$$

   Suppose that the publisher is interested in estimating the mean number of misprints per page λ based on n randomly selected pages from the book with number of misprints $X_1, \ldots, X_n$, respectively.

      i. Write down expressions for the mean and variance of $\bar{X} = \sum_{i=1}^{n} X_i / n$ and show that $\bar{X}$ is the uniformly minimum variance unbiased estimator (UMVUE) of λ. [7 marks]

      ii. Obtain the UMVUE of the variance of $\bar{X}$. [3 marks]

      iii. Suppose $n = 5$ and $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, $X_4 = 0$ and $X_5 = 2$. Compute the UMVU estimate of λ and that of the variance of $\bar{X}$. [2 marks]

5. For describing the annual household incomes of high-earning families in a certain community, a truncated Pareto distribution with the following probability density function (pdf)

$$f(x, \theta) = \theta x^{-2} I(x \geq \theta),$$

is used, where $\theta > 0$ is an unknown parameter.

(a) Sketch the likelihood function and find the maximum likelihood estimator (MLE) of θ. [6 marks]

(b) Derive the cumulative distribution function of the MLE and find its mean. Is the MLE unbiased for θ? If not, construct an unbiased estimator of θ based on the MLE. [7 marks]

(c) Prove that the MLE converges in probability to θ. [7 marks]

6. Let $X_1, \cdots, X_n$ be independent with $X_i \sim N(i\mu, 1)$, $1 \leq i \leq n$. Supose we want to test $H_0 : \mu = 0$ against $H_1 : \mu = \mu_0$, where $\mu_0 > 0$ is known. In the following problems you may need to use $\sum_{i=1}^n i = n(n+1)/2$ and $\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$.

(a) Use the Neyman-Pearson Lemma to show that the critical region of the most powerful test is of the form $\{\sum_{i=1}^n iX_i > k\}$.

Find the value of k corresponding to the most powerful size α test. [7 marks]

(b) Obtain a suitable pivotal quantity to construct a two-sided $100(1-\alpha)\%$ confidence interval for μ. [7 marks]

(c) Suppose $n = 4$ and the observed values are $X_1 = -0.5, X_2 = 0.4, X_3 = 1.1, X_4 = 3.5$. In the following you may need to use $P[N(0,1) > 1.645] = 0.05$ and $P[N(0,1) > 1.96] = 0.025$.

Use the critical region of (a) to test $H_0 : \mu = 0$ against $H_1 : \mu = 1$ at 5% level based on the above data. What is your conclusion?

Use the pivotal quantity of (b) to obtain a two-sided 95% confidence interval for μ based on the above data. [6 marks]

7. Suppose that there are 4 coins in a box, each one being either silver or copper. Let θ denote the total number of silver coins in the box so that the possible values of θ are $\{0, 1, 2, 3, 4\}$. Assume an ignorance prior for θ; this means $P[\theta = i] = 1/5$, $0 \leq i \leq 4$.

A coin is chosen at random from the box and suppose that it has turned out to be silver.

(a) Obtain the likelihood function $P[\text{silver coin}|\theta]$, for $\theta \in \{0, 1, 2, 3, 4\}$. [7 marks]

(b) Use the likelihood function and the ignorance prior to show that the posterior probabilities are given by

$$P[\theta = i|\text{silver coin}] = \frac{i}{10}, \quad 0 \leq i \leq 4.$$

[7 marks]

(c) Find the mean of the posterior distribution and hence find the Bayes estimator of the total number of silver coins in the box. [4 marks]

(d) Let Y denote the total number of copper coins in the box. What will be the Bayes estimator of Y given that when a coin is chosen at random from the box it turned out to be silver. [2 marks]