RESTRICTED OPEN BOOK EXAMINATION

Full marks can be obtained for complete answers to **four** questions.

Credit will only be given for the best **four** answers.

Candidates may bring into the examination hall up to six A4 pages of notes, written, typed or printed on both sides; but they must attach these notes to their examination scripts at the end of the examination.

Useful Information

1) For a random variable, X, with variance σ^2, Chebyshev's inequality states that for any $t > 0$

$$P\{\,|X - E(X)| > t\,\} \le \frac{\sigma^2}{t^2}.$$

2) If $X_1, \ldots, X_n$ are independent identically distributed random variables with probability density function $f(x)$ and distribution function $F(x)$, and if

$$X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$$

denote the corresponding order statistics, the probability density function, $p_j(y)$, of $Y = X_{(j)}$ is given by

$$p_j(y) = \frac{n!}{(j-1)!\,(n-j)!}\{F(y)\}^{j-1}\{1 - F(y)\}^{n-j}\; f(y) \qquad (j = 1, \ldots, n).$$

3) For any $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\, e^{-t}\, dt = (\alpha - 1)!.$$

4) If X has a Gamma distribution with parameters α and β, its probability density function is, with $\alpha > 0,\ \beta > 0$,

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}, \qquad 0 < x < \infty.$$

5) If a random variable X is Poisson with parameter $m > 0$, its probability mass function is:

$$f(x) = P(X = x) = e^{-m}\,\frac{m^x}{x!} \qquad (x = 0, 1, \ldots)$$

and $E(X) = m,\ V(X) = m$.

6) If χ_k^2 is chi-squared distributed with k degrees of freedom, $k \ge 1$, its probability density function is,

$$f(x|k) = 2^{-\frac{1}{2}k}\,\{\Gamma(\tfrac{1}{2}k)\}^{-1}\, x^{\frac{1}{2}k-1}\, e^{-\frac{1}{2}x}, \qquad 0 < x < \infty$$

and $E(\chi_k^2) = k,\ V(\chi_k^2) = 2k$.

1. A random variable X is distributed as Pareto with parameter α, that is, with probability density function

$$f(x) = \begin{cases} \dfrac{\alpha}{x^{\alpha+1}} & x > 1, \\ 0 & x \le 1, \end{cases}$$

where $\alpha > 2$.

If $R(x) = P(X > x)$, show that

$$R(x) = \begin{cases} x^{-\alpha} & \text{if } x > 1, \\ 1 & x \le 1. \end{cases}$$ [3 marks]

Find the mean, $E(X)$, and variance, $V(X)$, of X. [3 marks]

Apply Chebyshev's inequality to find an upper bound on the probability of the event

$$\left(|X - E(X)| > t \right)$$

for all $t > 0$. [3 marks]

Demonstrate that for all $t > (\alpha - 1)^{-1}$, the bound given by Chebyshev's inequality may be expressed as

$$P\left(X > t + \frac{\alpha}{\alpha - 1} \right) \le \frac{V(X)}{t^2}.$$

[6 marks]

If $\alpha = 3$, compare the exact probability of the event

$$\left(X > t + \frac{3}{2} \right)$$

with the bound given by Chebyshev's inequality for this probability and explain why, now, the Chebyshev bound is increasingly less sharp as t increases. [4 marks]

For $\alpha = 3$, evaluate the Chebyshev bound for the probabilities of the following events:

a) $(X > 6)$,

b) $(X > 9)$,

together with the actual probabilities of these two events. [4 marks]

In the light of your results, assess the usefulness of Chebyshev's inequality for providing a bound on the upper tail of the Pareto distribution. [2 marks]

2. A random variable, X, has an Exponential distribution with mean φ, that is, with probability density function

$$f(x) = \begin{cases} \frac{1}{\varphi} e^{-(x/\varphi)} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

where $\varphi > 0$ is unknown.

Find the distribution function, $F(x) = P(X \leq x)$ of X for all $x \in (-\infty,\infty)$. [2 marks]

Show that $V(X) = \varphi^2$. [3 marks]

Let $x = [x_1, \ldots, x_n]$ denote a vector consisting of a random sample of n observations from an Exponential distribution with mean φ. Write down the likelihood function and demonstrate that

$$S = \sum_{i=1}^{n} X_i$$

is sufficient for φ, where $X_1, \ldots, X_n$ are independent Exponential random variables, each with mean φ. [5 marks]

Show that

$$\bar{X}_n = \frac{S}{n},$$

provides an unbiased estimator of φ and write down its variance. [3 marks]

Suppose that $n = 3$. Let

$$X_{(1)} \leq X_{(2)} \leq X_{(3)}$$

denote the ordered random variables and let

$$M = X_{(2)}$$

be the random variable corresponding to the median of the observed sample. Show that the probability density function, $p_M(m)$, say, of M is given by

$$p_M(m) = \begin{cases} \frac{6}{\varphi} e^{-\frac{2m}{\varphi}} \left(1 - e^{-\frac{m}{\varphi}}\right) & m \geq 0, \\ 0 & m < 0. \end{cases}$$

Find the expected value, $E(M)$ of M and show that $\hat{\varphi}_M = \frac{6}{5} M$ provides an unbiased estimator of φ. [10 marks]

Explain why $\bar{X}_n$ may be preferred to $\hat{\varphi}_M$ as an estimator or φ. [2 marks]

3. Let $L = L(x|\theta)$ denote the likelihood function of a random sample of n observations from a distribution $f(x|\theta)$ and let T denote an unbiased estimator of $\tau(\theta)$, a function of θ, and put $U = \partial\,\text{logL}/\partial\theta$.

Prove that, under appropriate conditions,

i) $E(U) = 0$; [4 marks]

ii) $E(U^2) = -E(\partial^2 logL/\partial\theta^2)$; [3 marks]

iii) $cov(T,U) = \tau'(\theta) = \partial\tau(\theta)/\partial\theta$. [4 marks]

Hence derive the Cramer-Rao bound for the variance of T, namely

$$\text{V}(T) \geq -\{\tau'(\theta)\}^2 / E(\partial^2\text{logL}/\partial\theta^2).$$ [3 marks]

A random sample of n observations is available from the distribution with probability density function

$$f(x|\theta) = \begin{cases} (2\theta)^{-1}|x|^{\{(1-\theta)/\theta\}} & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that a minimum variance bound estimator of $\tau(\theta) = -\theta$ exists for this distribution. Find this estimator and give its variance. [8 marks]

Does a minimum variance bound estimator exist also for $\tau(\theta) = \theta$? If so, find this estimator and give its variance. [3 marks]

4. For testing the efficacy of a prototype kitchen mixer, a large number of raisins were dropped into a cake mix, which was then allowed to bake for an appropriate length of time. After the cake was fully cooked, it was cut into slices of equal size. Let X denote the number of raisins in a randomly selected slice. Under the hypothesis that the new gadget is an effective kitchen mixer, the raisins will be randomly distributed throughout the cake and X may be postulated to follow a Poisson distribution with probability mass function

$$f(x|\theta) = e^{-\theta}\,\frac{\theta^x}{x!} \qquad (x = 0, 1, \ldots),$$

where $\theta > 0$ is an unknown parameter of the distribution.

It is required to estimate $\tau(\theta) = e^{-\theta}$, where $\tau(\theta) = P(X = 0)$.

Verify that $f(x|\theta)$ is a member of the exponential family of distributions, that is, $f(x|\theta)$ may be written as

$$f(x|\theta) = a(x)\,b(\theta)\exp\{c(\theta)\,u(x)\},$$

where $a(x)$ and $u(x)$ are some functions of x and $b(\theta)$ and $c(\theta)$ are some functions of θ, and give explicit expressions for these functions. [3 marks]

Let $x_1, \ldots, x_n$ denote the observed number of raisins in n randomly selected slices of the cake, $n > 1$, and suppose that x_i is an observed value of X_i, $i = 1, \ldots, n$, where the random variables $X_1, \ldots, X_n$ are independent identically distributed with common probability function $f(x|\theta)$, as above. Deduce that

$$T = \sum_{i=1}^{n} X_i$$

is sufficient for θ, and state the distribution of T.

[3 marks]

A banal estimator of θ is

$$h(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } X_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Demonstrate that $h(X_1, \ldots, X_n)$ is unbiased for $\tau(\theta)$ and explain giving a reason whether or not you would recommend it for estimating $\tau(\theta)$. [3 marks]

Show that the conditional probability that $X_1 = 0$, conditional on $T = t$, is given by

$$P(X_1 = 0 \mid T = t) = \frac{(n-1)^t}{n^t}.$$

[5 marks]

Question 4 continued overleaf

Q4 contd.

Hence show that

$$g(T) = E\{h(X_1, ..., X_n) | T\} = \left(1 - \frac{1}{n}\right)^T.$$ [2 marks]

What properties does *g(T)* possess as an estimator of $\tau(\theta)$, according to the Rao-Blackwell theorem? [3 marks]

Demonstrate that

$$V\{g(T)\} = e^{-2\theta}\{e^{\theta/n} - 1\}.$$

[5 marks]

Explain why *g(T)* also provides the minimum variance unbiased estimator of $\tau(\theta)$.

[1 mark]

5. During peak periods on each week day, motor vehicles arrive at a busy traffic junction according to Poisson process at the rate of λ vehicles per time unit. For estimating λ, the value of *T*, the time taken from the start of the observation process until the *k*th motor vehicle arrives at the junction is recorded, where *T* has a gamma distribution with probability density function

$$f_T(t|\lambda) = \begin{cases} \dfrac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t}, & t > 0, \\ 0 & t \le 0, \end{cases}$$

and $k > 1$ is a fixed integer.

Demonstrate that the maximum likelihood estimator of λ based on an observed value, t, say, of T is given by

$$\hat{\lambda}_B = \frac{k}{T}.$$

[5 marks]

Show that

$$E(\hat{\lambda}_B) = \left(\frac{k}{k-1}\right)\lambda, \qquad k > 1$$

and deduce that $\hat{\lambda}_B$ provides a biased estimator of λ. [6 marks]

For $k = 2$ and 10, evaluate the bias of $\hat{\lambda}_B$ and comment on its implication in estimating λ.

[2 marks]

Suggest an unbiased estimator of λ based on T and evaluate its variance. [5 marks]

An alternative method of data collection is to record the value of K, where K denotes the number of motor vehicles arriving at the traffic junction in a fixed amount of time, t, say, and K has a Poisson distribution with mean λt, that is,

$$P\{K = k\} = f_K(k|\lambda) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad (k = 0, 1, \ldots,).$$

Suppose that $K = k$ is observed. Demonstrate that the maximum likelihood estimator of λ based on this data is given by $\hat{\lambda}_P$ where

$$\hat{\lambda}_P = \frac{K}{t}.$$

Show that $\hat{\lambda}_P$ is unbiased for λ with variance

$$V(\hat{\lambda}_P) = \frac{\lambda}{t}.$$

[5 marks]

Comment on why $\hat{\lambda}_P$ and $\hat{\lambda}_{UB}$ are strictly not comparable.

[2 marks]

6. The lifelength, X, of a certain electronic component is modelled by a Weibull distribution with probability density function

$$f(x) = \begin{cases} \theta^{-2} \, x \exp\left(-x^2 / 2\theta^2\right) & 0 \le x < \infty, \\ 0 & x < 0, \end{cases}$$

where $\theta > 0$ is an unknown parameter of the distribution.

Verify that $Y = X^2/\theta^2$ has a χ^2 distribution with 2 degrees of freedom. [4 marks]

Given a random sample of n observations, $x_1, \ldots, x_n$, from this distribution, prove that the most powerful test of the hypothesis

$$H_0: \ \theta = \theta_0$$

against the alternative

$$H_1: \ \theta = \theta_1 > \theta_0$$

has the critical region

$$\sum_{i=1}^{n} x_i^{\,2} > c_0 \, \theta_0^{\,2},$$

where c_0 is such that

$$P\left(\chi^2_{2n} > c_0\right) = \alpha$$

and α denotes the size of the test. [14 marks]

[N.B. You may use without proof, but should state explicitly, a standard distributional result on the sum of independent χ^2 random variables.]

Deduce that the test is uniformly most powerful against all alternative hypotheses of the form $H_1: \ \theta > \theta_0$. [2 marks]

If $n = 10$, $\theta_0 = 1$, $\theta_1 = 1.7$, find the critical region of size 0.05, and show that the power of the test is approximately 95%. [5 marks]

7. A computer program written by a trainee generates random numbers from a uniform distribution with probability density function

$$f(x|\theta) = \begin{cases} \theta^{-1} & 0 \le x \le \theta, \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta > 0$ is unknown.

A Bayesian statistician testing the computer program obtains a random sample of n random numbers, $x_1, \ldots, x_n$, generated by the program. Write down the likelihood function of the data. Suppose that x_i denotes the observed value of a random variable X_i $(i = 1, \ldots, n)$. Show that

$$X_{(n)} = \max(X_1, \ldots, X_n)$$

is sufficient for θ. [5 marks]

The statistician specifies the following prior distribution for θ:

$$\pi_0(\theta) = \begin{cases} 2\theta & 0 \le \theta \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value, d_0, say, of θ such that the prior probability that θ is less than d_0 is 0.95, that is, d_0 is such that

$$P(\theta < d_0) = 0.95.$$ [2 marks]

Evaluate also the mean, $E(\theta)$, of the prior distribution of θ. [1 mark]

Show that the posterior distribution, $\pi_1(\theta|x)$, of θ, conditional on data $x = [x_1, \ldots, x_n]$, is given by, with $n > 2$,

$$\pi_1(\theta|x) = \begin{cases} \left[\dfrac{(n-2)\{x_{(n)}\}^{n-2}}{1-\{x_{(n)}\}^{n-2}} \right] \dfrac{1}{\theta^{n-1}}, & x_{(n)} \le \theta \le 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_{(n)} = \max(x_1, \ldots, x_n)$. [7 marks]

Explain why $\pi_1(\theta|x)$ depends on x only through the observed value, $x_{(n)}$, of the sufficient statistic $X_{(n)}$ for θ. [3 marks]

If $n = 5$ and $x = [0.1, 0.2, 0.25, 0.45, 0.5]$, find the value, d_1, say, of θ such that, conditional on x, the posterior probability that θ is less than d_1 is 0.95, that is,

$$P\{(\theta|x) < d_1\} = 0.95.$$ [3 marks]

Question 7 continued overleaf

Q7 contd.

Find also the mean, $E(\theta|x)$, of the posterior distribution of θ. [2 marks]

Compare the values of d_0 and d_1 and of $E(\theta)$ and $E(\theta|x)$ and comment on how, in the light of the data, the statistician's beliefs concerning θ have changed. [2 marks]