1.

(i) Give the definition of the Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$. For any $a > 0$, find the Fourier transform $\widehat{f}(\xi)$ of

$$f(x) = \frac{1}{x^2 + a^2}.$$

[*Hint:* You will need to consider separately the cases $\xi \geq 0$ and $\xi \leq 0$, and you can use a semicircular contour in the lower half-plane if $\xi \geq 0$, and in the upper half-plane if $\xi \leq 0$. You need only do one of these cases if you can use the fact that f is real-valued to show that $\hat{f}(-\xi) = \overline{\widehat{f}(\xi)}$.]

(ii) Hence, or otherwise, find the Fourier transform of $g : \mathbf{R} \to \mathbf{C}$, where

$$g(x) = \frac{1}{(x^2+1)(x^2+4)}.$$

[20 marks]

2.

a) Determine whether the following functions are integrable, naming any results that you use.

$$f_1(x) = e^{-x^2},$$
$$f_2(x) = e^{ix}e^{-x^2},$$
$$f_3(x) = x^{-1}e^{-x^2}.$$

b) State Tonelli's Theorem. Now consider the function $F : \mathbf{R}^2 \to \mathbf{C}$ given by

$$F(x, y) = f(x - y)g(y)e^{-inx}\chi_{(-\pi,\pi)}(x)\chi_{(-\pi,\pi)}(y),$$

for any 2π-periodic functions f, $g : \mathbf{R} \to \mathbf{C}$ which are integrable on $(-\pi, \pi)$. Show that for any integer n, both double integrals of F are equal to $\widehat{f}(n)\widehat{g}(n)$, where $\widehat{f}(n)$ and $\widehat{g}(n)$ are the Fourier coefficients.

[20 marks]

3.

(i) For $-2\pi < y < 2\pi$, let

$$g(y) = \frac{y}{2\sin\frac{y}{2}}, \quad h(y) = \frac{1}{2\sin\frac{y}{2}} - \frac{1}{y}$$

Show that $g(0)$, $h(0)$ can be defined so that g, h are continuous functions on $(-2\pi, 2\pi)$

Now consider the function f defined by

$$f(x) = \begin{matrix} 1+x & \text{if } 0 \le x \le \pi, \\ -1 & \text{if } -\pi < x < 0, \end{matrix}$$

and extended 2π-periodically to a function on $\mathbf{R}$.

As usual, let $s_n(y)$ be defined for y not an integer multiple of 2π by

$$s_n(y) = \frac{1}{2\pi}\frac{\sin((n+\frac{1}{2})y)}{\sin(\frac{1}{2}y)},$$

and let

$$S_n(f)(x) = \int_{x-\pi}^{x+\pi} f(x-y)s_n(y)dy.$$

[This is the same as the usual formula, because the integrand is 2π-periodic.]

(ii) Show that, for $0 < x < \pi$,

$$S_n(f)(x) = -\frac{1}{\pi}\int_{x-\pi}^{x} g(y)\sin((n+\tfrac{1}{2})y)dy$$

$$+\frac{1}{\pi}\left((1+x)\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right)\left(h(y)+\frac{1}{y}\right)\sin((n+\tfrac{1}{2})y)dy$$

The Fourier Series Theorem says that $\lim_{n\to\infty} S_n(f)(x)$ exists for all x and gives the value of the limit: state this limit for this f and any $x \in (0,\pi)$.

(iii) Now let $x_n = \pi/(n+\frac{1}{2})$. *Assume that* uniformly for $-\frac{3}{2}\pi \le a < b \le \frac{3}{2}\pi$

$$\lim_{n\to\infty}\int_a^b g(y)\sin((n+\tfrac{1}{2})y)dy = \lim_{n\to\infty}\int_a^b h(y)\sin((n+\tfrac{1}{2})y)dy = 0$$

and that

$$\lim_{\Delta\to+\infty}\int_0^{\Delta}\frac{\sin y}{y}dy$$

exists. Show that

$$\lim_{n\to\infty} S_n(f)(x_n) = \frac{2}{\pi}\int_0^{\pi}\frac{\sin y}{y}dy.$$

[20 marks]

4. For $z = x + iy = re^{i\theta}$, let

$$P(r,\theta) = u(x,y) = \frac{1}{2\pi}\mathrm{Re}\left(\frac{1+z}{1-z}\right).$$

(i) Show that for $(x,y) \neq (1,0)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

You may find it helpful to use the fact that u is the real part of a holomorphic function. If so, explain briefly how you use this.

From now on *assume* that equation (1) is equivalent to

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} = 0. \tag{2}$$

(ii) Show that for $re^{i\theta} \neq 1$

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{|1-re^{i\theta}|^2}.$$

Verify that, for $0 \leq r < 1$,

$$P(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}.$$

Hence or otherwise, show that, for $0 < r < 1$

$$\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1.$$

(iii) Now let $f : \mathbf{R} \to \mathbf{C}$ be continuous and 2π-periodic. Show that F satisfies (2) with F replacing P for $r < 1$, where

$$F(r,\theta) = \int_{-\pi}^{\pi} f(t)P(r,\theta - t)dt.$$

Show also that

$$F(r,\theta) - f(\theta) = \int_{-\pi}^{pi}(f(\theta - t) - f(\theta)P(r,t)dt.$$

[20 marks]

5.

(i) Let $g : \mathbf{R} \to \mathbf{C}$ be integrable and $g_{a,b}(x) = g((x-a)/b)$, where $a \in \mathbf{R}$ and $b > 0$. Find the Fourier transform $\widehat{g}_{a,b}$ in terms of $\widehat{g}$.

(ii) Now suppose that $f : \mathbf{R} \to \mathbf{C}$ is integrable, that

$$u = u(x,t) : \mathbf{R} \times [0, \infty) \to \mathbf{C}$$

is continuous and locally uniformly integrable in x, that all first and second partial derivatives are defined and continuous on $\mathbf{R} \times (0, \infty)$ and locally uniformly integrable in x, and that they satisfy the equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + u, \tag{3}$$

$$u(x,0) = f(x). \tag{4}$$

Let $\widehat{u}(\xi, t)$ denote the Fourier transform of $u(x,t)$ with respect to x. Write down the Fourier transforms of (3) and (4). You need not justify your answer. By solving the resulting differential equation and boundary condition for $\widehat{u}(\xi, t)$, show that

$$\widehat{u}(\xi, t) = e^{t + i\xi t - \xi^2 t} \widehat{f}(\xi),$$

and hence or otherwise find an expression for $u(x,t)$, stating any general results that you use.

[*Hint*: You may assume that the Fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi} e^{-\xi^2/2}$.]

(iii) State the Dominated Convergence Theorem. Using the sequence of functions $y \mapsto (1/2\sqrt{\pi n}) f(y) e^{-(x-y+n)^2/4n}$ for positive integers n, or otherwise, show that

$$\lim_{n \to +\infty} e^{-n} u(x,n) = 0.$$

[20 marks]

6. Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{matrix} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 1. \end{matrix}$$

(i) Show that

$$\widehat{f}(\xi) = 2\frac{1 - \cos\xi}{\xi^2}.$$

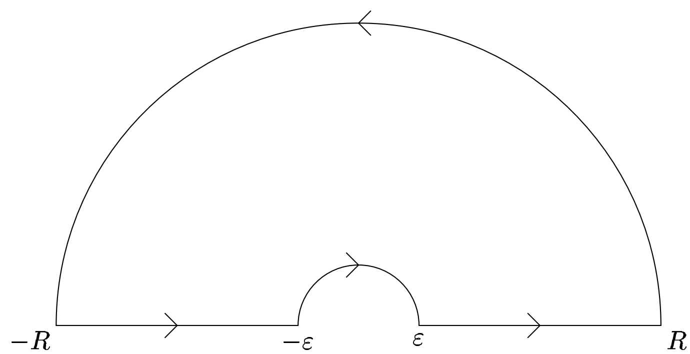
Show also that $\widehat{f}$ is integrable on $\mathbf{R}$.

(ii) Use an Inverse Fourier Theorem to evaluate

$$\int_{-\infty}^{\infty} e^{iyt}\frac{1 - \cos t}{t^2}dt$$

for all $y \in \mathbf{R}$. For $y < -1$, work out this integral directly by using the beehive contour drawn and the function

$$\frac{e^{iyz} - \frac{1}{2}e^{iyz+iz} - \frac{1}{2}e^{iyz-iz}}{z^2}.$$



In particular, explain why the integral round the large semicircle $\to 0$ as $R \to \infty$ for $y < -1$, and why the integral round the small semicircle $\to 0$ as $\varepsilon \to 0$.

[20 marks]

7.

(i) Fix $a \in (0, \infty)$. Let $f : (0, \infty) \to \mathbf{C}$ be any function such that $f(x)e^{-ax} \in L^1(0, \infty)$. Define the Laplace transform $\mathcal{L}f(z)$ for any z with $\text{Re}(z) \geq a$.

(ii) Show that if $f \in L^1(0, \infty)$ then

$$|\mathcal{L}(f)(z)| \leq \int_0^\infty |f(x)| dx$$

for all z with $\text{Re}(z) \geq 0$.

(iii) Show that if $f \in L^2(0, \infty)$ then $\mathcal{L}(f)(z)$ is defined for all z with $\text{Re}(z) > 0$, quoting any results that you use.

(iv) State Plancherel's Theorem, and use this and the connection between the Laplace transform and the Fourier transform to show that if $f \in L^2(0, \infty)$ then

$$\int_{-\infty}^\infty |\mathcal{L}(f)(t + iy)|^2 dy \leq \int_0^\infty |f(x)|^2 dx$$

for all $t > 0$.

(v) Work out $\mathcal{L}(\chi_{(a,b)})(z)$ for any finite interval $(a, b) \subset \mathbf{R}$. Hence, or otherwise, verify that

$$\frac{1 - e^{-z}}{z}$$

is the Laplace transform of a function in $L^1(0, \infty) \cap L^2(0, \infty)$.

Show however that $1/(z - i)$ is not the Laplace transform of any function in $L^1(0, \infty)$ or $L^2(0, \infty)$.

[*Hint*: (ii) and (iv) might be helpful.]

[20 marks]

8.

(i) Define the mean and variance of any probability measure μ for which

$$\int_{-\infty}^{\infty} x^2 d\mu(x) < +\infty. \tag{5}$$

Also, define the Fourier transform $\widehat{\mu}$ for any probability measure μ.

(ii) Find the mean and variance, if defined, of the following probability measures μ_1, μ_2.

a) $\mu_1(\{1\}) = \mu_1(\{-1\}) = \frac{1}{4}$, $\mu_1(\{0\}) = \frac{1}{2}$.

b) μ_2 is the probability measure with density function

$$\frac{1}{\pi}\frac{1}{1+x^2}.$$

(iii) Let μ be any probability measure for which (5) holds. Show that for all ξ, $h \in \mathbf{R}$ with $h \neq 0$,

$$\left|\frac{\widehat{\mu}(\xi+h)-\widehat{\mu}(\xi)}{h} + \int_{-\infty}^{\infty} ixe^{-i\xi x} d\mu(x)\right| \leq \int_{-\infty}^{\infty}\left|\frac{e^{-ihx}-1}{h} + ix\right| d\mu(x).$$

Give the values of $(d/d\xi)\widehat{\mu}(\xi)$ and $(d^2/d\xi^2)\widehat{\mu}(\xi)$, without any further proof.

(iv) Find $\widehat{\mu}_1$ and $\widehat{\mu}_2$ for μ_1 and μ_2 as in (ii), $j = 1, 2$. For μ_2, you might find it helpful to compute $\widehat{g}$ for $g(x) = e^{-|x|}$ and use an Inverse Fourier Theorem. Verify that $\widehat{\mu}_2$ is not differentiable everywhere.

[20 marks]