1. Give the definition of the Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$. Find the Fourier transform $\hat{f}(\xi)$ of

$$f(x) = \frac{1}{(x^2+9)^2}.$$

[*Hint:* You will need to consider separately the cases $\xi \geq 0$ and $\xi \leq 0$, and you can use a semicircular contour in the lower half-plane if $\xi \geq 0$, and in the upper half-plane if $\xi \leq 0$. You need only do one of these cases if you can use the fact that f is real-valued to show that $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$.]

[20 marks]

2. (i) Determine whether the following functions are integrable on the given domains, naming any theorems that you use.

a) $f(x) = \dfrac{1}{x}$ on $[1, \infty)$.

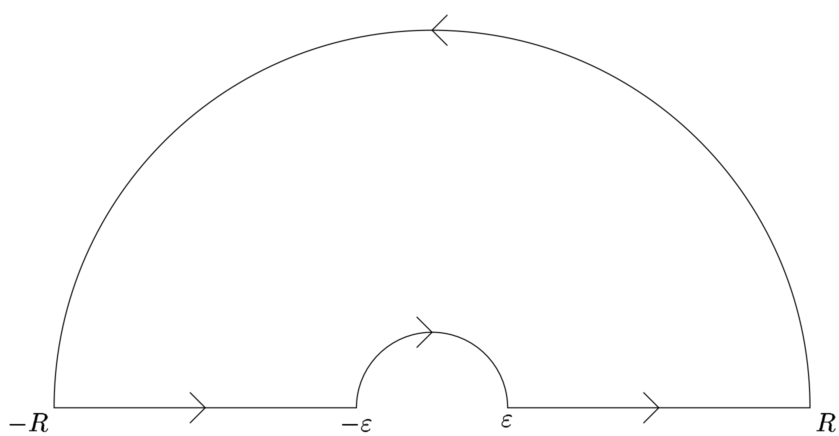
b)

$$f(x) = \frac{1}{(|x|+1)^2} \text{ on } \mathbf{R}.$$

c) $f(x) = \dfrac{\sin(x)}{x}$ on $(0, 1)$.

(ii) Use a semicircular "beehive" contour (shown below) to evaluate

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(x)}{x} dx.$$



[20 marks]

3. Consider the function defined by

$$g(x) = -x + \pi$$

for $x \in [0, 2\pi)$, and extended 2π-periodically to a function on $\mathbf{R}$.

As usual, let $s_n(y)$ be defined for y not an integer multiple of 2π by

$$s_n(y) = \frac{1}{2\pi}\frac{\sin((n+\frac{1}{2})y)}{\sin(\frac{1}{2}y)},$$

and let

$$S_n(g)(x) = \int_{x-\pi}^{x+\pi} g(x-y)s_n(y)dy.$$

[This is the same as the usual formula, because the integrand is 2π-periodic.]

(i) Show that

$$S_n(g)(x) = -x + \int_{x-\pi}^{x+\pi} ys_n(y)dy + \pi\left(\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right) s_n(y)dy.$$

You may assume that the integral of s_n over any interval of length 2π is 1.

The Fourier Series Theorem says that $\lim_{n\to\infty} S_n(g)(x)$ exists for all x and gives a value for the limit: state this limit for this g and for any $x \in (0, \pi)$.

(ii) Let

$$T_n(g)(x) = \left(\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right)\frac{\sin((n+\frac{1}{2})y)}{y}dy.$$

Show that if $x_n = \pi/(n+\frac{1}{2})$ then

$$\lim_{n\to\infty} T_n(g)(x_n) = 2\int_0^{\pi}\frac{\sin y}{y}dy.$$

Assuming (as is true) that

$$\lim_{n\to\infty}(S_n(g)(x) - T_n(g)(x)) = 0$$

uniformly in x, and that

$$\int_0^{\pi}\frac{\sin y}{y}dy > \frac{\pi}{2},$$

explain why the convergence of $S_n(g)(x)$ to its limit cannot be uniform on $(0, \pi)$.

[20 marks]

4. Suppose that

$$u = u(x,t) : [0, 2\pi] \times [0, \infty) \to \mathbf{C}$$

is a continuous function, and that the partial derivatives u_t, u_x, u_{xx} exist on $(0, 2\pi) \times (0, \infty)$, with u_x, u_{xx} extending continuously to $[0, 2\pi] \times [0, \infty)$. Suppose also that

$$u(0,t) = u(2\pi, t) = u_x(0,t) = u_x(2\pi, t) = 0$$

for all $t \geq 0$. Consider the equation

$$u_t = u_{xx}, t > 0, 0 < x < 2\pi, \qquad (1)$$

with initial condition

$$u(x, 0) = f(x). \qquad (2)$$

As usual, define the Fourier coefficients

$$\hat{u}(n,t) = \int_0^{2\pi} u(x,t) e^{-inx} dx,$$

and define similarly the Fourier coefficients of u_x, u_{xx}, u_t.

(i) Find formulae for $\hat{u}_x(n,t)$ and $\hat{u}_{xx}(n,t)$ in terms of $\hat{u}(n,t)$. Derive from (1) and (2) a differential equation involving $\hat{u}(n,t)$ and $(d/dt)\hat{u}(n,t)$, explaining any theory that you use. Hence show that the Fourier series of $u(x,t)$ with respect to x is given by

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{-n^2 t} \hat{f}(n) e^{inx},$$

where $\hat{f}(n)$ are the Fourier coefficients of $f(x)$.

(ii) Now suppose that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < +\infty.$$

Show that for some constant C,

$$\left| u(x,t) - \frac{1}{2\pi} \hat{f}(0) \right| \leq C e^{-t},$$

and also that, for all integers N,

$$|u(x,t) - f(x)| \leq C N^2 t \sum_{n=-N}^{N} |\hat{f}(n)| + C \sum_{|n|>N} |\hat{f}(n)|. \qquad (3)$$

[*Hint*: You may assume that, for all real $y \geq 0$, $|e^{-y} - 1| \leq \text{Max}(1, y)$.]

Deduce from (3) that

$$\lim_{t \to 0} u(x,t) = f(x).$$

[20 marks]

5. (i) Let $\hat{f}(\xi)$ be the Fourier transform of $f(x) = e^{-x^2/2}$. Using the rectangular contour drawn below, show that

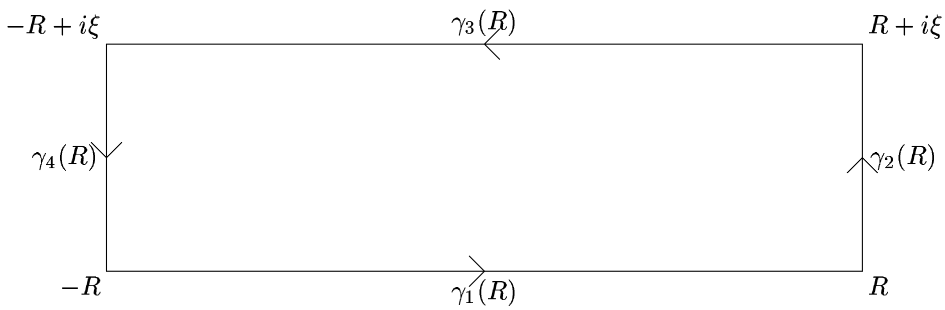
$$\lim_{R\to\infty} \int_{\gamma_3(R)} e^{-z^2/2} dz = -e^{-\xi^2/2} \hat{f}(\xi)$$

and

$$\lim_{R\to\infty} \int_{\gamma_2(R)} e^{-z^2/2} dz = \lim_{R\to\infty} \int_{\gamma_4(R)} e^{-z^2/2} dz = 0.$$

Hence, or otherwise, calculate $\hat{f}(\xi)$. You may assume that $\hat{f}(-\xi) = \hat{f}(\xi)$, and that

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}.$$



In what follows, you may assume that, for any real number a,

$$|e^{-ia} - 1 + ia| \leq 3\text{Min}(|a|, |a|^2).$$

(ii) Now let $f : \mathbf{R} \to \mathbf{C}$ be a function such that $f,\ g \in L^1(\mathbf{R})$, where $g(x) = xf(x)$. Show that

$$\left| \frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h} + i\hat{g}(\xi) \right| \leq 3 \int_{-\infty}^{\infty} \text{Min}(|x|, x^2|h|)|f(x)|dx.$$

By breaking up the integral on the right into the pieces $|x| \leq |h|^{-1/3}$ and $|x| \geq |h|^{-1/3}$, or otherwise, show that $\hat{f}$ is differentiable and

$$\frac{d}{d\xi}\hat{f}(\xi) = -i\hat{g}(\xi).$$

(iii) Hence, or otherwise, find the Fourier transforms of the functions $xe^{-x^2/2}$ and $x^2e^{-x^2/2}$.

[20 marks]

6. (i) State Tonelli's Theorem about double integrals.

(ii) Let f, g, $\widehat{g} \in L^1(\mathbf{R})$. Show that

$$(\widehat{f}\widehat{g})^{\vee} = f * (\widehat{g})^{\vee}.$$

[*Hint:* Try writing $(\widehat{f}\widehat{g})^{\vee}$ as a double integral involving f and $\widehat{g}$.]

(iii) Now let f be continuous, bounded and integrable. For $t > 0$, let

$$g_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}.$$

Show that

$$\int_{-\infty}^{\infty} g_t(y)dy = 1$$

for all $t > 0$, and that, for all $\delta > 0$,

$$\lim_{t \to 0} \int_{|y| \geq \delta} g_t(y)dy = 0.$$

[*Hint: You may assume* that

$$\int_{-\infty}^{\infty} g_1(y)dy = 1.]$$

(iv) Hence, using (ii), show that

$$f(x) - \frac{(\widehat{f}\widehat{g}_t)^{\vee}(x)}{2\pi} = \int_{-\infty}^{\infty} (f(x) - f(x-y))g_t(y)dy$$

and

$$f(x) = \lim_{t \to 0} \frac{(\widehat{f}\widehat{g}_t)^{\vee}(x)}{2\pi}.$$

[*Hint*: You may assume that $(\widehat{g}_t)^{\vee} = 2\pi g_t$. You are *not* required to work out $\widehat{g}_t$ or $(\widehat{g}_t)^{\vee}$.]

[20 marks]

7. Let $f \in L^1(0,\infty)$.

(i) Define the Laplace transform $\mathcal{L}(f) : \{z \in \mathbf{C} : \mathrm{Re}(z) > 0\} \to \mathbf{C}$. Show that $\mathcal{L}(f)$ is bounded. Show also that

$$\lim_{\mathrm{Re}(z)\to+\infty} \mathcal{L}(f)(z) = 0.$$

[*Hint*: You may need to use the Dominated Convergence Theorem for a family of functions parametrised by the positive reals.]

Assuming (as is true) that for any $A > 0$,

$$\lim_{\mathrm{Im}(z)\to\infty} \mathcal{L}(f)(z) = 0$$

uniformly for $0 < \mathrm{Re}(z) \leq A$, show that

$$\lim_{z\to\infty, \mathrm{Re}(z)>0} \mathcal{L}(f)(z) = 0.$$

(ii) Determine which of the following can be the Laplace transform of a function in $L^1(0,\infty)$, giving brief reasons. For any which can, find $f_i \in L^1(0,\infty)$ such that $F_i = \mathcal{L}(f_i)$.

a) $F_1(z) = \dfrac{1}{|z+1|}$.

b) $F_2(z) = e^{-z^2}$.

c) $F_3(z) = \dfrac{1}{z+1}$.

d) $F_4(z) = \dfrac{1}{z^2-1}$.

[20 marks]

8. In this question, you may assume that the function

$$\frac{1}{\sqrt{\pi}}e^{-x^2/4}$$

has integral 1 on $(-\infty, \infty)$, and has Fourier transform $e^{-\xi^2}$.

(i) Define the *mean* and the *variance* of a probability measure. Now let

$$f(x) = \frac{1}{2}e^{-|x|}.$$

Check that the integral of f over $\mathbf{R}$ is 1. Compute the mean and variance for the probability measure μ with density function f.

(ii) For μ as in (i), compute the Fourier transform $\hat{\mu}$. Now let $(*)^n\mu$ denote the n-fold convolution of μ, and let μ_n be the measure on $\mathbf{R}$ defined by

$$\mu_n(A) = \int_{-\infty}^{\infty} \chi_A(x/\sqrt{n})d(*^n\mu).$$

Show that

$$\hat{\mu}_n(\xi) = (\hat{\mu}(\xi/\sqrt{n}))^n.$$

Hence, or otherwise, show that for any fixed ξ

$$\lim_{n\to\infty} \ln \hat{\mu}_n(\xi) = -\xi^2.$$

Relate this to what the Central limit Theorem says about

$$\lim_{n\to\infty} \int g(x)d\mu_n(x)$$

for any bounded measurable function g on $\mathbf{R}$.

[20 marks]