1.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$$

[2 marks] Standard theory.

Now let

$$f(x) = \frac{1}{x^2 - 2x + 2}.$$

To compute the Fourier transform, we consider the function

$$f(z) = \frac{e^{-iz\xi}}{z^2 - 2z + 2}.$$

[2 marks]

Let $\xi \geq 0$. If $\mathrm{Im}(z) \leq 0$ then $|e^{-iz\xi}| = e^{\mathrm{Im}(z)\xi} \leq 1$. So let $\gamma_R = \gamma_1(R) \cup \gamma_2(R)$ be the anticlockwise contour in the lower half plane, with $\gamma_1(R)$ being the straightline from R to $-R$ and $\gamma_2(R)$ being the semicircle arc. We have $|z^2 - 2z + 2| \geq |z|^2 - 2|z| - 2$. So

$$|f(z)| \leq \frac{1}{R^2 - 2R - 2} \text{ for } z \in \gamma_2(R).$$

[4 marks]

So

$$|\int_{\gamma_2(R)} \frac{e^{-iz\xi}}{z^2 - 2z + 2} dz| \leq \frac{\pi R}{R^2 - 2R - 2} \to 0 \text{ as } R \to \infty.$$

[2 marks]

We have

$$z^2 - 2z + 2 = (z - 1 - i)(z - 1 + i) = 0$$

if and only if $z = 1 \pm i$. So the only singularity of f inside γ_R is at $1 - i$. So

$$\int_{\gamma_R} f(z) dz = 2\pi i \mathrm{Res}(f(z), 1 - i) = \frac{2\pi i e^{-\xi - i\xi}}{-2i} = -\pi e^{-\xi - i\xi}.$$

[4 marks]

So

$$\hat{f}(\xi) = -\lim_{R \to \infty} \int_{\gamma_1(R)} f(z) dz$$

$$= -\lim_{R \to \infty} \int_{\gamma(R)} f(z) dz = \pi e^{-\xi - i\xi}.$$

[2 marks]

Now since $f(x)$ is real for real x,

$$\hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$$

$$= \overline{\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx} = \overline{\hat{f}(\xi)}.$$

[3 marks]

So for all real ξ, we have

$$\hat{f}(\xi) = \pi e^{-|\xi|-i\xi}.$$

[1 mark]

$2+2+4+2+4+2+3+1 = 20$ marks. Similar to homework exercises.

2.(i) *Tonelli's Theorem* Let $f : \mathbf{R}^2 \to \mathbf{C}$ be Lebesgue measurable and suppose that one of the double integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|dxdy, \ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|dydx$$

is finite. Then the functions

$$x \mapsto \int_{-\infty}^{\infty} f(x,y)dy, \ y \mapsto \int_{-\infty}^{\infty} f(x,y)dx$$

are both finite a.e, and the two double interals are both finite, and are equal.
[5 marks] Standard theory from lectures.
(ii)a)

$$\int_0^{\infty}\int_0^{\infty} xe^{-x^2(1+u^2)}dxdu = \int_0^{\infty} \lim_{R\to\infty}\left[\frac{-1}{2(1+u^2)}e^{-x^2(1+u^2)}\right]_0^R du$$

$$= \int_0^{\infty} \frac{1}{2(1+u^2)}du = \lim_{R\to\infty}\left[\frac{1}{2}\arctan(u)\right]_0^R = \frac{\pi}{4}.$$

[4 marks]
However, putting $xu = y$, $xdu = dy$, and so

$$\int_0^{\infty}\int_0^{\infty} xe^{-x^2-x^2u^2}dudx = \int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}dydx = \int_0^{\infty} e^{-x^2}dx\int_0^{\infty} e^{-y^2}dy,$$

as required.
[3 marks]
By Tonelli's theorem these two integrals are equal. But the second one is equal to

$$\left(\int_0^{\infty} e^{-x^2}dx\right)^2.$$

So we deduce that

$$\int_0^{\infty} e^{-x^2}dx = \sqrt{\pi/4} = \frac{1}{2}\sqrt{\pi}.$$

[2 marks] This example was done in lectures. Some examples on using Tonelli's Theorem were set.
(ii)b) We have

$$\int_{-\infty}^{\infty} |f*g(x)|dx = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(x-y)g(y)dy\right|dx \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)||g(y)|dydx,$$

using $|f(x-y)g(y)| = |f(x-y)||g(y)|$. By Tonelli's Theorem, this is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(y)||f(x-y)|dxdy = \int_{-\infty}^{\infty} |g(y)|\int_{-\infty}^{\infty} |f(u)|dudy,$$

using the change of variable $u = x - y$, $du = dx$ on the inner integral. But then this is equal to

$$\int_{-\infty}^{\infty} |g(y)| dy \int_{-\infty}^{\infty} |f(u)| du.$$

[4 marks]

This is finite. So both the repeated integrals are finite and equal, and, again by Tonelli, the single integral $f * g$ is finite a.e..

[2 marks] This example was done in lectures. Some examples on using Tonelli's Theorem were set.

$5 + 4 + 3 + 2 + 4 + 2 = 20$ marks.

3(i) If $x < y < x + \pi$ then $-\pi < x - y < 0$, and $g(x - y) = x - y - 1$. If $x - \pi < y < x$ then $0 < x - y < \pi$ and $g(x - y) = x - y + 1$.
[3 marks] So if, as usual, we write

$$s_n(y) = \frac{\sin((n + \frac{1}{2})y)}{2\pi \sin(\frac{1}{2}y)},$$

$$S_n(g)(x) = \int_{x-\pi}^{x+\pi} (x - y)s_n(y)dy - \left(\int_x^{x+\pi} - \int_{x-\pi}^{x}\right) s_n(y).$$

[1 mark]

Since the integral of s_n over any interval of length 2π is 1, we obtain

$$S_n(g)(x) = x - \int_{x-\pi}^{x+\pi} ys_n(y)dy - \left(\int_x^{x+\pi} - \int_{x-\pi}^{x}\right) s_n(y)dy.$$

[3 marks]

The Fourier Series Theorem says that for each x

$$\lim_{n\to\infty} S_n(g)(x) = \frac{1}{2}(g(x+) + g(x-)) = x + 1 \text{ for } 0 < x < \pi.$$

[2 marks]
(ii) Make the change of variable $u = (n + \frac{1}{2})y$. Then $du = (n + \frac{1}{2})dy$ and $dy/y = du/u$. When $y = x_n$ then $u = \pi$ and when $y = x_n \pm \pi$, $u = \pi \pm (n + \frac{1}{2})\pi$. So

$$T_n(g)(x_n) = -\frac{1}{\pi}\left(\int_\pi^{\pi(n+1+\frac{1}{2})} - \int_{\pi(\frac{1}{2}-n)}^{\pi}\right) \frac{\sin u}{u} du.$$

[3 marks]

Now $(\sin u)/u$ is an even function and

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin u}{u} du = 2 \lim_{R\to\infty} \int_0^R \frac{\sin u}{u} du$$

exists. So

$$\lim_{n\to\infty} T_n(g)(x_n) = -\frac{1}{\pi} \lim_{R\to\infty} \int_\pi^R \frac{\sin u}{u} du + \frac{1}{\pi} \lim_{R\to\infty} \int_{-R}^{\pi} \frac{\sin u}{u} du = \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} du.$$

[3 marks]

Now if convergence of $S_n(g)(x)$ to its limit is uniform then the limit is $1 + x$ for all $x \in (0, \pi)$, and given $\epsilon > 0$, there is N such that for all $n \geq N$ and all $x \in (0, \pi)$,

$$|S_n(x) - x - 1| < \epsilon.$$

But

$$\lim_{n\to\infty} T_n(g)(x_n) = \frac{2}{\pi} \int_0^\pi \frac{\sin y}{y} dy = 1 + a$$

for $a > 0$ we have. So taking $\epsilon = \frac{1}{2}a$ and any n such that $x_n < \frac{1}{2}a$ we get a contradiction.
[5 marks]
$3 + 1 + 3 + 2 + 3 + 3 + 5 = 20$ marks. Similar to homework exercise.

4 (i)a) Change of order of integration is allowed since the integrand is continuous and we are integrating over a rectangle. We have

$$\hat{h}(n) = \int_{-\pi}^{\pi} e^{-inx} \int_{-\pi}^{\pi} f(x-y)g(y)dydx = \int_{-\pi}^{\pi} g(y) \int_{-\pi}^{\pi} f(x-y)e^{-inx}dxdy$$

Make the variable change $u = x - y$, $du = dx$ on the inner integral. The inner integral limits change to $-\pi - y$ and $\pi - y$, but we can change them back again to $-\pi$ and π because the integrand is 2π-periodic. So we have

$$\hat{h}(n) = \int_{-\pi}^{\pi} g(y) \int_{-\pi}^{\pi} f(u)e^{-inu-iny}dudy = \hat{g}(n)\hat{f}(n).$$

[5 marks]
(i)b) By integration by parts, we have

$$\hat{f}_1(n) = \int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \left[f(x)e^{-inx}\right]_{-\pi}^{\pi} + in\int_{-\pi}^{\pi} f(x)e^{-inx}dx = in\hat{f}(n),$$

because f is 2π-periodic, so $f(\pi) = f(-\pi)$.
[3 marks]

Then since $f_1' = f_2$, we have $\hat{f}_2(n) = (in)^2\hat{f}(n) = -n^2\hat{f}(n)$.
[1 mark]
(ii)a) By (i) applied to the function $\theta \to u(r,\theta)$, the Fourier coefficients of $\partial^2 u/\partial\theta^2$ are $-n^2\hat{u}(r,n)$. General theory says that the Fourier coefficients of $\partial u/\partial r$ are obtained by differentiating

$$\int_{-\pi}^{\pi} u(r,\theta)e^{-in\theta}d\theta$$

with respect to r, giving $(d/dr)\hat{u}(r,n)$. Similarly the Fourier coefficients of $\partial^2 u/\partial r^2$ are $(d/dr)^2\hat{u}(r,n)$. So taking Fourier coefficients in the p.d.e. we obtain

$$\frac{d^2}{dr^2}\hat{u}(r,n) + \frac{1}{r}\frac{d\hat{u}}{dr}(r,n) - \frac{n^2}{r^2}\hat{u}(r,n) = 0. \tag{1}$$

[4 marks]
(ii)b) We try a solution r^m to (1) and we find that

$$m(m-1)r^{m-2} + mr^{m-2} - n^2r^{m-2} = 0.$$

Then $m(m-1) + m - n^2 = m^2 - n^2 = 0$and $m = \pm n$. So the general solution is $A_nr^n + B_nr^{-n}$ for constant A_n and B_n if $n \neq 0$.
[3 marks]

If $n = 0$ this method only gives one linearly independent solution. But we can get a second by direct calulation. We have

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\hat{u}(r,0) = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\hat{u}(r,\theta)\right) = 0.$$

So for a constant B_0,

$$\frac{d}{dr}\hat{u}(r,\theta) = \frac{B_0}{r}$$

and

$$\hat{u}(r,\theta) = B_0 \log r + A_0.$$

[4 marks]
$3+2+3+1+4+3+4 = 20$ marks. Standard theory. Homework exercise set on working out the details of solving Laplace' equation in the complement of the unit disc.

5(i)a) Making the change of variable $x/a = t$, $dx/a = dt$, we have $x = at$ and

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x/a)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f(t)e^{-ita\xi}adt = a\hat{f}(a\xi).$$

[2 marks]
(i)b) Making the change of variable $x + a = t$, $dx = dt$, $x = t - a$, so

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} f(x+a)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f(t)e^{iat-it\xi}dt = e^{ia\xi}\hat{f}(\xi).$$

[2 marks]
(ii) Try $g(x) = f(x/a)$ and $k(x) = g(x+d) = f((x+d)/a) = e^{-(x+d)^2/2a^2}$. Then

$$\hat{k}(\xi) = ae^{-i\xi d}\hat{f}(a\xi) = \sqrt{2\pi}ae^{-i\xi d - a^2\xi^2/2}.$$

So try $a = \sqrt{2b}$ and $d = +c$. Then $\sqrt{2\pi}a = 2\sqrt{\pi b}$. So the required function is

$$\frac{1}{2\sqrt{b\pi}}e^{-(x+c)^2/4b}.$$

[3 marks]
(iii) The Fourier transforms of $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ are

$$\frac{\partial \hat{u}}{\partial t}(\xi,t),\ i\xi\hat{u}(\xi,t),\ -\xi^2\hat{u}(\xi,t).$$

[3 marks]
So the transforms of the p.d.e. and the boundary condition are

$$\frac{\partial \hat{u}}{\partial t} = (-i\xi - \xi^2)\hat{u}(\xi,t),$$

$$\hat{u}(\xi,0) = \hat{f}(\xi).$$

[2 marks]
The solution of this o.d.e. with respect to t is

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-it\xi - t\xi^2}.$$

[3 marks]
By (ii), $e^{-it\xi - t\xi^2}$ is the Fourier transform of $(1/2\sqrt{\pi t})e^{-(x-t)^2/4t}$.
[2 marks]
The product of Fourier transforms is the Fourier transform of a convolution, and any Fourier transform of an integrable function is the Fourier transform of just one function. So

$$\hat{u}(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-y)e^{-(y-t)^2/4t}dy.$$

[3 marks]
$2 + 2 + 3 + 3 + 2 + 3 + 2 + 3 = 20$ marks. Similar to homework exercises.

6(i) Putting $u = x^2/4t$ gives $1/\sqrt{t} = 2\sqrt{u}/|x|$.

So

$$0 \leq \lim_{t\to 0} \varphi_t(x) = \frac{1}{|x|\sqrt{\pi}} \lim_{u\to+\infty} \sqrt{u}e^{-u} \leq \frac{1}{|x|\sqrt{\pi}} \lim_{u\to+\infty} \frac{u}{e^u} = 0.$$

It is possible to use the version of l'Hopital's Rule at ∞ on the last limit
[4 marks]

By the change of variable $u = x/\sqrt{t}$, $x^2/4t = u^2/4$. and $du = dx/\sqrt{t}$. So

$$\int_{-\infty}^{\infty} \varphi_t(x)dx = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2/4}du = 1,$$

[2 marks and

$$\int_{|x|\geq\delta} \varphi_t(x)dx = \frac{1}{2\sqrt{\pi}} \int_{|u|\geq\delta/\sqrt{t}} e^{-u^2/4}du \to 0 \text{ as } t \to 0.$$

[3 marks]
(ii)a) Let $|f(x)| \leq M$ for all x. Then

$$|\varphi_t * f(x)| \leq \int_{-\infty}^{\infty} |\varphi_t(x-y)f(y)|dy$$

$$\leq M \int_{-\infty}^{\infty} \varphi_t(x-y)dy = M \int_{-\infty}^{\infty} \varphi_t(u)du = M$$

for all x and t, using the change of variable $u = x - y$, $-dy = du$ for the middle equality on the last line.
[3 marks] Similar to homework exercises up to here.
(ii)b) Making the change of variable $x - y = u$, so that $y = x - u$, $-dy = du$, and using the fact that φ_t has integral 1,

$$|\varphi_t*f(x)-f(x)| \leq \int_{-\infty}^{\infty} \varphi_t(u)|f(x-u)-f(x)|du = \left(\int_{-\delta}^{\delta} + \int_{|u|\geq\delta}\right) \varphi_t(u)|f(x-u)-f(x)|du.$$

[2 marks]

Now $|f(x-u) - f(x)| \leq 2M$ for all x and u. Given $\epsilon > 0$ and x, choose $\delta > 0$ so that if $|y| \leq \delta$ then $|f(x-y) - f(x)| \leq \epsilon/2$. Then for this fixed δ, choose t_0 so that for all $t \leq t_0$,

$$\int_{|u|\geq\delta} \varphi_t(u)du < \epsilon/4M.$$

Then for $t \leq t_0$,

$$|\varphi_t * f(x) - f(x)| \leq \frac{\epsilon}{2} \int_{-\delta}^{\delta} \varphi_t(u)du + 2M \int_{|u|\geq\delta} \varphi_t(u)du < \frac{\epsilon}{2} + \frac{2M\epsilon}{4M} = \epsilon.$$

[6 marks] Standard theory - with hints provided - for these last 8 marks.
$4 + 2 + 3 + 3 + 2 + 6 = 20$ marks.

7.(i)

$$\mathcal{L}(f)(z) = \int_0^\infty f(x)e^{-xz}dx.$$

[1 mark]

Write $z = t + iu$ for t and u real. Then

$$|e^{-xz}| = |e^{-xt-ixu}| = e^{-xt} \leq 1$$

for $x \geq 0$ and $t \geq 0$. So for $\mathrm{Re}(z) \geq 0$,

$$|\mathcal{L}(f)(z)| \leq \int_0^\infty |f(x)e^{-xz}|dx \leq \int_0^\infty |f(x)|dx = \|f\|_1,$$

and so $\mathcal{L}(f)(z)$ is bounded.

[3 marks]

We have

$$\left|\frac{\mathcal{L}(f)(z+h) - \mathcal{L}(f)(z)}{h} + \int_0^\infty xe^{-xz}f(x)dx\right| \leq$$

$$|h|\int_0^\Delta x^2e^{-x\mathrm{Re}(z)/2}|f(x)|dx + \int_\Delta^\infty xe^{-x\mathrm{Re}(z)/2}|f(x)|dx$$

Now

$$\lim_{x\to\infty} xe^{-x\mathrm{Re}(z)/2} = 0, \ \lim_{x\to\infty} x^2e^{-x\mathrm{Re}(z)/2} = 0$$

(by writing these as quotients with $e^{x\mathrm{Re}(z)/2}$ in the quotient and using l'Hopital's Rule as $x \to \infty$, for example). So there is $M > 0$ such that, for all $x \geq 0$,

$$|xe^{-x\mathrm{Re}(z)/2}| \leq M, \ |x^2e^{-x\mathrm{Re}(z)/2}| \leq M.$$

which means that

$$|h|\int_0^\Delta x^2e^{-x\mathrm{Re}(z)/2}|f(x)|dx \leq |h|M\int_0^\infty |f(x)|dx < \frac{\epsilon}{2}$$

if h is sufficiently small, and

$$\int_\Delta^\infty xe^{-x\mathrm{Re}(z)/2}|f(x)|dx \leq M\int_\Delta^\infty |f(x)| < \frac{\epsilon}{2}$$

if Δ is sufficiently large, because $f \in L^1(0,\infty)$. So if Δ is sufficiently large given ϵ and h is sufficiently small given z and Δ and ϵ

$$\left|\frac{\mathcal{L}(f)(z+h) - \mathcal{L}(f)(z)}{h} + \int_0^\infty xf(x)e^{-xz}dx\right| < \epsilon.$$

This implies that $\mathcal{L}(f)$ is holomorphic with derivative $\mathcal{L}(xf(x))$, as required.

[6 marks] Standard theory, somewhat reduced and with hints provided, as far as here.
(iii)a) $F_1 = \mathcal{L}(f_1)$ where $f_1(x) = e^{-ax} \in L^1(0, \infty)$.
[2 marks]
(iii)b) Since F_2 has a singularity at a, it cannot be holomorphic on the right half-plane and cannot be $\mathcal{L}(f)$ for any $s \in L^1(0, \infty)$¿ (in fact, it is $\mathcal{L}(f)$ for $f(x) = e^{ax}$ which is not in $L^1(0, \infty)$.)
[2 marks]
(iii)c) F_3 is not holomorphic on any open set: it is real-valued with nonconstant real part, and so cannot satisfy the Cauchy-Riemann equations.
[3 marks]
(iii)d) We have $|F_4(z)| = e^{-\mathrm{Im}(z)}$ and $-\mathrm{Im}(z)$ is not bounded above on the right half plane. So F_4 is not bounded above and cannot be $\mathcal{L}(f)$ for any $f \in L^1(0, \infty)$.
[3 marks] Partly unseen, but some similar problems set for these last 10 marks.
$1 + 3 + 6 + 2 + 2 + 3 + 3 = 20$ marks.

8(i) a) The mean m and variance σ are given by

$$m = \frac{1}{2}2 + \frac{1}{2}(-2) = 0, \ \sigma = \frac{1}{2}2^2 + \frac{1}{2}(-2)^2 = 4.$$

[3 marks]

(i)b) The mean is 0 because the function

$$\frac{xe^{-x^2/8}}{2\sqrt{2\pi}}$$

is odd and hence has integral 0. The variance σ is given by

$$\sigma = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/8}}{2\sqrt{2\pi}} = \lim_{R\to\infty} \left[\frac{-2xe^{-x^2/8}}{\sqrt{2\pi}} \right]_{-R}^{R} + \int_{-\infty}^{\infty} \frac{2e^{-x^2/8}}{\sqrt{2\pi}} dx = 4,$$

because the limits of both upper and lower values of the square bracket term are 0.
[4 marks]
(ii)

$$\hat{\mu}(\xi) = \frac{1}{2}e^{-2i\xi} + \frac{1}{2}e^{2i\xi}.$$

[1 mark]. So

$$(\hat{\mu})^n(\xi) = 2^{-n}(e^{-2i\xi} + e^{2i\xi})^n = 2^{-n}e^{-2ni\xi}(1 + e^{4i\xi})^n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} e^{i(4k-2n)\xi}.$$

Now $(\hat{\mu})^n(\xi)$ is the Fourier transform of $*^n\mu$. So we see that for integers k with $0 \le k \le n$,

$$*^n\mu(\{4k - 2n\}) = 2^{-n} \binom{n}{k}.$$

[3 marks] Standard homework exercises thus far. Now we have

$$\hat{\mu}_n(\xi) = \int_{-\infty}^{\infty} e^{-i\xi/\sqrt{n}} d *^n \mu = (\hat{\mu}(\xi/\sqrt{n}))^n.$$

[2 marks] Standard theory

Now

$$\hat{\mu}(\xi/\sqrt{n}) = \frac{1}{2}(e^{-2i\xi/\sqrt{n}} + e^{2i\xi/\sqrt{n}})$$

$$= \frac{1}{2}(1 - \frac{2i\xi}{\sqrt{n}} - \frac{4\xi^2}{2n} + \frac{8i\xi^3}{n^{3/2}3!} + \cdots + 1 + \frac{2i\xi}{\sqrt{n}} - \frac{4\xi^2}{2n} - \frac{8i\xi^3}{n^{3/2}3!} + \cdots$$

$$= 1 - \frac{2\xi^2}{n} + \frac{16\xi^2}{4n^2} - \cdots$$

So

$$n \ln \hat{\mu}_n(\xi) = n\left(-\frac{2\xi^2}{n} + \frac{16\xi^2}{4!n^2} - \cdots\right) \to -2\xi^2 \text{ as } n \to \infty.$$

[4 marks] Unseen, but similar example done in class.

The normal density function with mean 0 and variance 4 (like μ in (i)a)) is that in (i)b). Then the Central Limit Theorem says that for all Lebesgue measurable A

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \chi_A(x) d\mu_n(x) = \int_{-\infty}^{\infty} \chi_A(x) \frac{e^{-8x^2}}{2\sqrt{2\pi}} dx.$$

[3 marks] Standard theory applied to a standard example.
$3+4+1+3+2+4+3 = 20$ marks.