1. Give the definition of the Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$. Find the Fourier transform $\hat{f}(\xi)$ of

$$f(x) = \frac{1}{x^2 - 2x + 2}.$$

[*Hint:* You will need to consider separately the cases $\xi \geq 0$ and $\xi \leq 0$, and you can use a semicircular contour in the lower half-plane if $\xi \geq 0$, and in the upper half-plane if $\xi \leq 0$. You need only do one of these cases if you can use the fact that f is real-valued to show that $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$.]

[20 marks]

2. (i) State Tonelli's Theorem.

(ii)a) Show that

$$\int_0^\infty \int_0^\infty x e^{-x^2 - x^2 u^2}\, dx du = \frac{\pi}{4}$$

and

$$\int_0^\infty \int_0^\infty x e^{-x^2 - x^2 u^2}\, du dx = \int_0^\infty e^{-y^2}\, dy \int_0^\infty e^{-x^2}\, dx.$$

Hence, or otherwise, compute

$$\int_0^\infty e^{-x^2}\, dx.$$

(ii)b) Show that if f and g are integrable on $\mathbf{R}$, then so is

$$f * g(x) = \int_{-\infty}^\infty f(x-y) g(y) dy,$$

with

$$\int_{-\infty}^\infty |f * g(x)| dx \leq \int_{-\infty}^\infty |f(x)| dx \int_{-\infty}^\infty |g(y)| dy.$$

[20 marks]

3. (i) Consider the function

$$g(x) = \begin{cases} 1+x & \text{if } x \in [0,\pi], \\ -1+x & \text{if } x \in (-\pi, 0), \end{cases}$$

and extend g to a 2π-periodic function on $\mathbf{R}$. As usual, let $s_n(y)$ be defined for y not an integer multiple of 2π by

$$s_n(y) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})y)}{\sin(\frac{1}{2}y)},$$

and let

$$S_n(g)(x) = \int_{x-\pi}^{x+\pi} g(x-y)s_n(y)dy.$$

[This is the same as the usual formula, because the integrand is 2π-periodic.] Show that

$$S_n(g)(x) = x - \int_{x-\pi}^{x+\pi} ys_n(y)dy - \left(\int_x^{x+\pi} - \int_{x-\pi}^{x} \right) s_n(y)dy.$$

You may assume that the integral of s_n over any interval of length 2π is 1.

The Fourier Series Theorem says that $\lim_{n\to\infty} S_n(g)(x)$ exists for all x and gives a value for the limit: state this limit for this g and for any $x \in (0,\pi)$.

(ii) Let

$$T_n(g)(x) = -\frac{1}{\pi} \left(\int_x^{x+\pi} - \int_{x-\pi}^{x} \right) \frac{\sin((n+\frac{1}{2})y)}{y} dy.$$

Show that if $x_n = \frac{\pi}{n+\frac{1}{2}}$ then

$$\lim_{n\to\infty} T_n(g)(x_n) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin y}{y} dy.$$

Assuming (as is true) that

$$\lim_{n\to\infty} (S_n(g)(x) - T_n(g)(x)) = 0$$

uniformly in x, and that

$$\int_0^{\pi} \frac{\sin y}{y} dy > \frac{\pi}{2}$$

explain why the convergence of $S_n(g)(x)$ to its limit cannot be uniform on $(0,\pi)$.

[20 marks]

4. (i) Let $f,\, g : \mathbf{R} \to \mathbf{C}$ be continuous and 2π-periodic. Let

$$f * g(\theta) = \int_0^{2\pi} f(\theta - t)g(t)dt.$$

a) Show that if $h = f * g$ then for all $n \in \mathbf{Z}$,

$$\hat{h}(n) = \hat{f}(n)\hat{g}(n),$$

where (as usual) $\hat{f}(n)$ denote the Fourier coefficients of f.

b) Show that if f has continuous derivatives $f_1 = f'$ and $f_2 = f''$, then

$$\hat{f}_1(n) = in\hat{f}(n) \text{ and } \hat{f}_2(n) = -n^2\hat{f}(n).$$

(ii) Let $u = u(r,\theta) : [0,1) \times \mathbf{R} \to \mathbf{C}$ be continuous, 2π-periodic in θ, and twice continuously differentiable in each of r and θ on $(0,1) \times \mathbf{R}$, and let

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Let $\hat{u}(r,n)$ $(n \in \mathbf{Z})$ denote the Fourier coefficients of $\theta \mapsto u(r,\theta)$.

a) Show that

$$\frac{d^2}{dr^2}\hat{u}(r,n) + \frac{1}{r}\frac{d}{dr}\hat{u}(r,n) - \frac{n^2}{r^2}\hat{u}(r,n) = 0. \qquad (1)$$

You should state any general results that you have not proved in (i)

b) Find the general solution to (1).

[20 marks]

5. (i) Let $f : \mathbf{R} \to \mathbf{C}$ be integrable.

a) Show that if $a > 0$ and $g(x) = f(x/a)$ then $\hat{g}(\xi) = a\hat{f}(a\xi)$.

b) Show that if $a \in \mathbf{R}$ and $h(x) = f(x+a)$ then $\hat{h}(\xi) = e^{ia\xi}\hat{f}(\xi)$.

(ii) Let $b,\ c \in \mathbf{R}$ with $b > 0$. Find the function whose Fourier transform is

$$e^{i\xi c}e^{-b\xi^2}.$$

You may use the fact that the Fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}e^{-\xi^2/2}$.

(iii) Now suppose that $u = u(x,t) : \mathbf{R} \times [0,\infty) \to \mathbf{C}$ is continuous and uniformly integrable in x, that all first and second partial derivatives are defined and continuous on $\mathbf{R} \times (0,\infty)$ and uniformly integrable in x, and that they satisfy the equations

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

$$u(x,0) = f(x). \qquad (2)$$

Let $\hat{u}(\xi,t)$ denote the Fourier transform of $u(x,t)$ with respect to x. Write down the Fourier transform of (1) and (2). You need not justify your answer. By solving the resulting differential equation and boundary condition for $\hat{u}(\xi,t)$, show that

$$\hat{u}(\xi,t) = e^{-i\xi t - \xi^2 t}\hat{f}(\xi),$$

and hence or otherwise find an expression for $u(x,t)$, stating any general results that you use.

[20 marks]

6. In this question, you may assume that

$$\int_{-\infty}^{\infty} e^{-x^2/4} dx = 2\sqrt{\pi}.$$

For $t > 0$, let

$$\varphi_t(x) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}.$$

(i) Show that if $x \neq 0$ then

$$\lim_{t \to 0} \varphi_t(x) = 0.$$

Show that

$$\int_{-\infty}^{\infty} \varphi_t(x) dx = 1,$$

and that, for any $\delta > 0$,

$$\lim_{t \to 0} \int_{|x| \geq \delta} \varphi_t(x) dx = 0.$$

[*Hint*: Use the change of variable $u = x/\sqrt{t}$ in both integrals.]

(ii) Let $f : \mathbf{R} \to \mathbf{C}$ be continuous, bounded and integrable and let

$$\varphi_t * f(x) = \int_{-\infty}^{\infty} \varphi_t(x - y) f(y) dy.$$

a) Show that $\varphi_t * f(x)$ is bounded uniformly in t and x.

b) Show, by breaking up the integral into two parts or otherwise, that

$$\lim_{t \to 0} \varphi_t * f(x) = f(x).$$

[*Hint*:

$$\varphi_t * f(x) - f(x) = \int_{-\infty}^{\infty} \varphi_t(y)(f(x - y) - f(x)) dy.$$

If you use this you should justify it. Write the integral as the sum over sets $\{y : |y| \leq \delta\}$ and $\{y : |y| \geq \delta\}$.]

[20 marks]

7. Let $f \in L^1(0,\infty)$.

(i) Define the Laplace transform $\mathcal{L}(f) : \{z \in \mathbf{C} : \mathrm{Re}(z) > 0\} \to \mathbf{C}$. Show that $\mathcal{L}(f)$ is bounded.

For the next part, you may assume that, for any $\Delta > 0$ and $h \in \mathbf{C}$ with $|h| < \Delta^{-1}$ and $|h| < \mathrm{Re}(z)/2$,

$$\left| \frac{\mathcal{L}(f)(z+h) - \mathcal{L}(f)(z)}{h} + \int_0^\infty x f(x) e^{-xz} dx \right| \leq$$

$$|h| \int_0^\Delta x^2 |f(x)| e^{-\mathrm{Re}(z)x/2} dx + \int_\Delta^\infty x |f(x)| e^{-\mathrm{Re}(z)/2} dx. \tag{1}$$

Using this or otherwise, show that $\mathcal{L}(f)(z)$ is holomorphic on $\{z \in \mathbf{C} : \mathrm{Re}(z) > 0\}$ with derivative

$$-\int_0^\infty x e^{-xz} f(x) dx.$$

[*Hint.* It suffices to show that, for a given z, the right hand side of (1) can be made arbitrarily small by taking Δ sufficiently large and h sufficiently small.]

(ii) Determine which of the following can be the Laplace transform of a function in $L^1(0,\infty)$. For any which can, find $f_i \in L^1(0,\infty)$ such that $F_i = \mathcal{L}(f_i)$.

a) $F_1(z) = \dfrac{1}{z+a}$ for $a \in \mathbf{C}$, $\mathrm{Re}(a) > 0$.

b) $F_2(z) = \dfrac{1}{z-a}$ for $a \in \mathbf{C}$, $\mathrm{Re}(a) > 0$.

c) $F_3(z) = e^{-|z|}$.

d) $F_4(z) = e^{iz}$.

[20 marks]

8. In this question, you may assume that the function

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2/8} \qquad (1)$$

has integral 1 on $(-\infty, \infty)$, and has Fourier transform $e^{-2\xi^2}$.

(i) Compute the mean and variance of the following probability measures on $\mathbf{R}$.

a) The measure μ defined by $\mu(\{2\}) = \mu(\{-2\}) = \frac{1}{2}$.

b) The measure λ with probability density function $f(x)$ as in (1).

(ii) Compute $\hat{\mu}(\xi)$ for μ as in (i)a), and compute $\hat{\mu}^n(\xi)$ for all integers $n \geq 2$. Hence, or otherwise, compute

$$*^n \mu(\{4k - 2n\})$$

where $*^n \mu$ is the n-fold convolution of μ and k is any integer with $0 \leq k \leq n$.

(iii) Let the probability measure μ_n on $\mathbf{R}$ be defined by

$$\mu_n(A) = \int_{-\infty}^{\infty} \chi_A(x/\sqrt{n})d(*^n \mu)$$

Show that

$$\hat{\mu}_n(\xi) = (\hat{\mu}(\xi/\sqrt{n}))^n.$$

Hence or otherwise show that for any fixed ξ

$$\lim_{n\to\infty} \ln \hat{\mu}_n(\xi) = -2\xi^2.$$

Relate this to what the Central limit Theorem says about

$$\lim_{n\to\infty} \mu_n(A)$$

for any Lebesgue measurable set $A \subset \mathbf{R}$.

[20 marks]