1. Give the definition of the Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$. Find the Fourier transform $\widehat{f}(\xi)$ of

$$f(x) = \frac{1}{(1+x^2)^3}.$$

[*Hint:* You will need to consider separately the cases $\xi \geq 0$ and $\xi \leq 0$, and you can use a semicircular contour in the lower half-plane if $\xi \geq 0$, and in the upper half-plane if $\xi \leq 0$. You need only do one of these cases if you can use the fact that f is real-valued to show that $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$.]

[20 marks]

2. (i) Find $\lim_{y\to 0} f(y)$ where

$$f(y) = \frac{1}{y} - \frac{1}{2\sin\left(\frac{1}{2}y\right)}.$$

Deduce that f extends to a continuous function on $[0,\pi]$.

(ii) Now consider the function

$$g(x) = \begin{cases} 1 \text{ if } x \in [0,\pi], \\ 0 \text{ if } x \in (-\pi,0), \end{cases}$$

and extend g to a 2π-periodic function on $\mathbf{R}$. As usual, let

$$S_n(g)(x) = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} g(x-y)\frac{\sin\left(\left(n+\frac{1}{2}\right)y\right)}{\sin\left(\frac{1}{2}y\right)}dy.$$

[This is the same as the usual formula, because the integrand is 2π-periodic.]

Show that if $0 < x < \pi$,

$$\lim_{n\to\infty}\left(S_n(g)(x) - \frac{1}{\pi}\left(\int_0^x + \int_0^{\pi-x}\right)\frac{\sin\left(\left(n+\frac{1}{2}\right)y\right)}{y}dy\right) = 0. \qquad (1)$$

[*Hint* : you may find it helpful to use part (i) and the Riemann Lebesgue Lemma.]

The Fourier Series Theorem says that $\lim_{n\to\infty} S_n(g)(x)$ exists for all x and gives a value for the limit: state this limit. By considering $x_n = \pi/(n+\frac{1}{2})$ or otherwise, show that the convergence is not uniform in x. You may assume that the convergence in (1) is uniform (which is true) and that the improper integral

$$\int_0^\infty \frac{\sin y}{y}dy = \frac{\pi}{2}.$$

[20 marks]

3. a) Show that $\dfrac{\sin x}{x}$ is integrable on $(0,\pi)$.

b) Show that $\dfrac{\sin x}{x}$ is not integrable on $(0,\infty)$, by considering $\left|\dfrac{\sin x}{x}\right|$ on $[(n+\frac{1}{4})\pi,(n+\frac{3}{4})\pi]$ or otherwise. Show, however, by using the fact that this is an even function and considering a contour integral of the function $\dfrac{e^{iz}}{z}$ (or otherwise) that

$$\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

[20 marks]

4. Let $f = f(\theta) : \mathbf{R} \to \mathbf{R}$ and $u = u(r,\theta) : [1,R] \times \mathbf{R} \to \mathbf{R}$ be continuous and 2π-periodic. Let $\partial u/\partial r$, $\partial^2 u/\partial r^2$, $\partial u/\partial \theta$, $\partial^2 u/\partial \theta^2$ be continuous for $1 < r < R$, $\theta \in \mathbf{R}$. As usual, let $\widehat{f}(n)$ denote the Fourier coefficients of f, and $\widehat{u}(r,n)$ the Fourier coefficients of $u(r,\theta)$ with respect to θ. Let u satisfy Laplace's equation in $1 < r < R$, $\theta \in \mathbf{R}$ (in polar coordinates) so that $\widehat{u}(r,n)$ satisfies the ordinary differential equation:

$$\frac{d^2\widehat{u}}{dr^2}(r,n) + \frac{1}{r}\frac{d\widehat{u}}{dr}(r,n) - \frac{n^2}{r^2}\widehat{u}(r,n) = 0. \tag{1}$$

Let the boundary conditions for u be

$$u(1,\theta) = f(\theta),$$
$$u(R,\theta) = 0.$$

(i) a) Give the boundary conditions on $\widehat{u}(1,n)$ and $\widehat{u}(R,n)$.

b) Show that

$$\widehat{u}(r,0) = \widehat{f}(0)(1 - (\ln r/\ln R)).$$

c) Now let $n \neq 0$, $n \in \mathbf{Z}$. Show that

$$\widehat{u}(r,n) = A_n r^{|n|} + B_n r^{-|n|}$$

for suitable A_n, B_n, and hence show that for a constant $C_1 > 0$ independent of R,

$$|\widehat{u}(r,n) - r^{-|n|}\widehat{f}(n)| \leq C_1 R^{-|n|}|\widehat{f}(n)|.$$

d) Show that

$$1 + \sum_{n=1}^{\infty} r^{-n}(e^{in\theta} + e^{-in\theta}) = \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta}|^2}.$$

(ii) Hence or otherwise show that, for a constant C_2 independent of R,

$$\left| u(r,\theta) + \frac{\ln r}{2\pi \ln R}\int_0^{2\pi} f(t)dt - \frac{1}{2\pi}\int_0^{2\pi} f(\theta - t)\frac{1 - r^{-2}}{|1 - r^{-1}e^{it}|^2}dt \right| \leq \frac{C_2}{R}\int_0^{2\pi} |f(t)|dt.$$

[20 marks]

5. (i) State Tonelli's Theorem about double integrals.

(ii) Let f, g, $\hat{g} \in L^1(\mathbf{R})$. Show that

$$(\hat{f}\hat{g})\check{} = f * \check{\hat{g}}$$

[*Hint:* Try writing $(\hat{f}\hat{g})\check{}$ as a double integral involving f and $\hat{g}$.]

(iii) Now let f be continuous, bounded and integrable. For $\lambda > 0$, let

$$g_\lambda(y) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + y^2}.$$

Show that

$$\lim_{\lambda \to 0} \int_{|y| \geq \delta} g_\lambda(y)dy = 0$$

for all $\delta > 0$ and that

$$\int_{-\infty}^{\infty} g_\lambda(y)dy = 1$$

for all $\lambda > 0$. Hence, using (ii), show that

$$f(x) - \frac{(\hat{f}\hat{g}_\lambda)\check{}(x)}{2\pi} = \int_{-\infty}^{\infty} (f(x) - f(x-y))g_\lambda(y)dy$$

and

$$f(x) = \lim_{\lambda \to 0} \frac{(\hat{f}\hat{g}_\lambda)\check{}(x)}{2\pi}.$$

You may assume that $(\hat{g}_\lambda)\check{} = 2\pi g_\lambda$.

[20 marks]

6. In this question, we shall say that a function $v = v(x,y) : \mathbf{R}\times(0,\infty) \to \mathbf{C}$ is *uniformly integrable in* x if

$$\sup_{y>0} \int_{-\infty}^{\infty} |v(x,y)|dx < +\infty,$$

$$\lim_{\Delta\to\infty} \int_{|x|\geq\Delta} |v(x,y)|dx = 0 \text{ uniformly in } y.$$

Let $u = u(x,y) : \mathbf{R}\times[0,\infty) \to \mathbf{R}$ be continuous and integrable in x for all $y \geq 0$. Let the partial derivatives u_x, u_{xx}, u_y, u_{yy} be defined and continuous for $(x,y) \in \mathbf{R}\times(0,\infty)$ and uniformly integrable in x. Let

$$u_{xx} + u_{yy} = 0 \text{ for all } x,\ y \in \mathbf{R} \text{ with } y > 0, \text{ and}$$

$$u(x,0) = f(x).$$

Let $\hat{u}(\xi,y)$ denote the Fourier transform of $u(x,y)$ with respect to x.

a) Give, without proof, the partial differential equation in y, and the boundary condition, satisfied by $\hat{u}(\xi,y)$. Show that

$$\hat{u}(\xi,y) = \hat{f}(\xi)e^{-|\xi|y}.$$

Hence give a formula for $u(x,y)$. You may assume that the inverse Fourier transform of $e^{-|\xi|y}$ with respect to ξ is $y/(\pi(x^2+y^2))$.

b) Show that for any $y > 0$,

$$\left|\int_{-\infty}^{\infty} e^{-i\xi x}\frac{u(x,y+h)-u(x,y)}{h}dx - \int_{-\infty}^{\infty} e^{-i\xi x}u_y(x,y)dx\right|$$
$$\leq \frac{1}{|h|}\int_{[0,h]}\int_{-\infty}^{\infty} |u_y(x,y+t)-u_y(x,y)|dxdt.$$

[*Hint.* You may use Tonelli's Theorem. This is the first step in the proof of the formula used in part a) for $\hat{u}_y(\xi,y)$, which leads to a similar expression for $\hat{u}_{yy}(\xi,y)$.]

[20 marks]

7. In this question, we say that $v = v(x,t) : (0,\ell)\times(0,\infty) \to \mathbf{C}$ is *uniformly integrable in* t if

$$\sup_{0<x<\ell} \int_0^\infty |v(x,t)|dt < \infty,$$

$$\lim_{\Delta\to+\infty} \int_\Delta^\infty |v(x,t)|dt = 0 \text{ uniformly in } x.$$

(i) Define the Laplace transform $\mathcal{L}(f)(z)$ of a function $f \in L^1(0,\infty)$ for $\operatorname{Re}(z) \geq 0$.

a) Show that if $a > 0$, and $g(t) = f(t-a)\chi_{[a,\infty)}(t)$,

$$\mathcal{L}(g)(z) = e^{-za}\mathcal{L}(f)(z).$$

b) Show also that if f is continuous on $[0,\infty)$ and the derivative f' exists and is continuous and integrable,

$$\mathcal{L}(f')(z) = z\mathcal{L}(f)(z) - f(0).$$

(ii) Now let $u = u(x,t) : [0,\ell] \times [0,\infty) \to \mathbf{R}$ be continuous, and integrable in t. Let the partial derivatives u_t, u_{tt} u_x, u_{xx} be defined and continuous on $(0,\ell) \times (0,\infty)$ and uniformly integrable in t.
Consider the equation

$$u_{tt} = u_{xx}\,,\ t > 0,\ 0 < x < \ell,$$
$$u(x,0) = 0 = \lim_{t\to 0} u_t(x,t),$$
$$u(\ell,t) = 0,\ u(0,t) = h(t)\ [\text{so that } h(0) = 0.]$$

Let $\mathcal{L}(u)(x,z)$ be the Laplace transform of $u(x,t)$ with respect to t.

Write down the differential equation in x satisfied by $\mathcal{L}(u)$. Show that

$$\mathcal{L}(u)(x,z) = \frac{\mathcal{L}(h)(z)e^{-zx}}{1-e^{-2z\ell}} - \frac{\mathcal{L}(h)(z)e^{(x-2\ell)z}}{1-e^{-2z\ell}}.$$

Expand this out as a power series and hence show that

$$u(x,t) = \sum_{n=0}^{\infty} h(t-x-2n\ell)\chi_{[x+2n\ell,\infty)}(t) - \sum_{n=1}^{\infty} h(t+x-2n\ell)\chi_{[2n\ell-x,\infty)}(t).$$

Deduce that for $0 \leq t \leq \ell$,

$$u(x,t) = \begin{cases} h(t-x) & \text{if } x \leq t \leq \ell, \\ 0 & \text{if } t \leq x. \end{cases}$$

[20 marks]

8. Let μ be any probability measure on $\mathbf{R}$.

a) Show that $\widehat{\mu}$ is a continuous function with $|\widehat{\mu}(\xi)| \leq 1$ for all ξ. You may use the facts that

$$\lim_{R \to +\infty} \left(\int_{-\infty}^{-R} + \int_{R}^{\infty} \right) d\mu = 0$$

and, for any continuous function f on any bounded interval $[a, b]$,

$$\left| \int_a^b f(x) d\mu(x) \right| \leq \text{Max}\{|f(x)| : x \in [a, b]\}.$$

b) If $n \geq 1$ is an integer, let $*^n \mu = \nu_n$ denote the n-fold convolution of μ and let μ_n be the measure such that

$$\mu_n(A) = \int \chi_A(x/\sqrt{n}) d(*^n \mu)(x).$$

State what the Central Limit Theorem says about μ_n for any probability measure μ with a certain property or properties (which you should also state). Give without proof the formula for $\widehat{\nu}_n(\xi)$ in terms of $\widehat{\mu}(\xi)$ and derive the formula for $\widehat{\mu}_n(\xi)$ in terms of $\widehat{\mu}(\xi)$.

c) Now let μ be the measure such that for any Borel-measurable set A,

$$\mu(A) = \frac{1}{\pi} \int \chi_A(x) \frac{dx}{1 + x^2}.$$

Find $\widehat{\mu}(\xi)$ by using a contour integral (you may cut down the work by using the fact that $\widehat{\mu}(\xi) = \widehat{\mu}(-\xi)$.) Hence find $\widehat{\mu}_n(\xi)$ for all n. Explain why the Central Limit Theorem fails for this measure.

[20 marks]