1. (a) *Stating a definition from lecture notes.* [5 marks]

Let $\tilde{f}$ be a lift of f, and $x \in \mathbb{R}$. The degree of f is $\tilde{f}(x+1) - \tilde{f}(x)$.

This makes sense since:

- The function $\tilde{f}(x+1) - \tilde{f}(x)$ is integer valued and continuous, so must be constant, hence independent of x.
- Any other lift of f is of the form $\tilde{f}_n(x) = \tilde{f}(x) + n$. Then

$$\tilde{f}_n(x+1) - \tilde{f}_n(x) = (\tilde{f}(x+1) + n) - (\tilde{f}(x) + n) = \tilde{f}(x+1) - \tilde{f}(x).$$

Therefore, the definition of the degree does not depend on the lift chosen.

(b) i. *Calculation similar to examples from lectures and homeworks.* [8 marks]

Let

$$F(z,t) = \frac{3z + it}{3 - itz}.$$

This is well-defined and continuous as long as the denominator is never zero.

$$3 - itz = 0 \Leftrightarrow 3 = itz \Rightarrow 3 = |itz| = |t|\,|z| = t$$

as $|z| = 1$. This cannot happen as $0 \leqslant t \leqslant 1$.

$|F(z,t)| = 1$ since

$$|3z + it| = |\bar{z}|\,|3z + it| = |3z\bar{z} + it\bar{z}| = |3 + it\bar{z}| = |\overline{3 - itz}| = |3 - itz| \neq 0,$$

So F is a homotopy from $f_0(z) = F(z,0) = z$ to $F(z,1) = f_1(z)$.

$$\deg(f_1) = \deg(f_0) = 1.$$

ii. *Calculation similar to examples from lectures and homeworks.* [7 marks]

Let

$$F(z,t) = \frac{t(z^3+2) - 4z^2}{|t(z^3+2) - 4z^2|}.$$

This is well-defined, continuous, and $|F(z,t)| = 1$ as long as $|t(z^3+2) - 4z^2|$ is never zero.

$$t(z^3+2) - 4z^2 = 0 \Leftrightarrow 4z^2 = t(z^3+2) \Rightarrow 4 = |4z^2| = |t(z^3+2)| \leqslant t(|z^3| + 2) = 3t,$$

which cannot occur as $t \leqslant 1$. So F is a homotopy from

$$f_0(z) = F(z,0) = \frac{-4z^2}{|-4z^2|} = -z^2 \text{ to } F(z,1) = \frac{z^3 + 2 - 4z^2}{|z^3 + 2 - 4z^2|} = f_1(z).$$

$$\deg(f_1) = \deg(f_0) = 2.$$

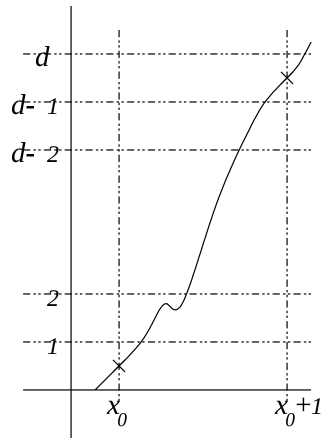
2. *A step-by-step leading through of a fairly abstract calculation. Similar to lecture result and homework question.*

(a) *From lectures.* [3 marks]
If $g_1(x) = k \in \mathbb{Z}$, then $\tilde{f}(x) = x + k$, so $f(e^{2\pi ix}) = e^{2\pi i\tilde{f}(x)} = e^{2\pi i(x+k)} = e^{2\pi ix}e^{2\pi ik} = e^{2\pi ix}$.
If $g_2(x) \in \mathbb{Z}$, then $e^{2\pi ix}$ is fixed by f^2, so is a point of period 2.

(b) *New calculation using familiar ideas.* [3 marks]
$g_2(x) = \tilde{f}^2(x) - x = \tilde{f}(x+k) - x = \tilde{f}(x) + kd - x = x + k + kd - x = k(d+1)$

(c) *Essentially a special case of result from lectures* [2 marks]
$g_2(x+1) - g_2(x) = (\tilde{f}^2(x+1) - (x+1)) - (\tilde{f}^2(x) - x) = \tilde{f}(\tilde{f}(x) + d) - x - 1 - \tilde{f}^2(x) + x = \tilde{f}^2(x) + d^2 - 1 - \tilde{f}^2(x) = d^2 - 1$.

(d) *Similar to lecture example and homework question.* [3 marks]
$d^2 - 1 < g_2(x_0 + 1) < d^2$.



(e) *Reproduce argument from lectures.* [3 marks]
Suppose $1 \leqslant< n \leqslant d^2 - 1$. Then $g_2(x_0) < n < g_2(x_0 + 1)$ so, by the Intermediate Value Theorem, there is a point $x_n \in (x_0, x_0 + 1)$ such that $g_2(x_n) = n$.

(f) *Tricky. General case to part of a homework problem.* [4 marks]
If $g_1(x_n) \in \mathbb{Z}$, then $n = k(d+1)$ for some $k \in Z$. Note that if $k = d - 1$, $k(d+1) = (d-1)(d+1) = d^2 - 1$. So $g_1(x_n) \in \mathbb{Z}$ for $n = (d+1), 2(d+1), \ldots (d-1)(d+1)$. Thus only $d - 1$ of these x_n can be solutions to $g_1(x) \in \mathbb{Z}$. Therefore, there are at least $d^2 - 1 - (d-1) = d^2 - d$ solutions of $g_2(x) \in \mathbb{Z}$ in $(x_0, x_0 + 1)$ which are not solutions of $g_1(x) \in \mathbb{Z}$.

(g) *Mimic argument from lectures.* [2 marks]
If x is a solution of $g_2(x) \in \mathbb{Z}$ but not of $g_1(x) \in \mathbb{Z}$, then $f^2(e^{2\pi ix}) = e^{2\pi ix}$ but $f(e^{2\pi ix}) \neq e^{2\pi ix})$, so $e^{2\pi ix}$ is a point of least period 2. Each of the points in $(x_0, x_0 + 1)$ gives a different point of S^1, so there are at least $d^2 - d = d(d-1)$ points of least period 2.

3. (a) *Stating result from lectures.* [4 marks]

If $\deg(f) = d$ and $|d| > 1$, then there is a degree-one semiconjugacy from f to $z \mapsto z^d$. If in addition, $|\tilde{f}'| > 1$ for any lift $\tilde{f}$ of f, then there is a conjugacy from f to $z \mapsto z^d$.

(b) i. *Standard calculation.* [3 marks]

$\deg(f_a) = \tilde{f}_a(1) - \tilde{f}_a(0) = (2 - \frac{1}{2} + 0) - (0 - \frac{1}{2} - 0) = 2$, so $d = 2$.

ii. *Standard calculation.* [3 marks]

$\tilde{f}'(x) = 2 + a\cos(2\pi x)$, so if $|a| < 1$, $\tilde{f}'(x) \geq 2 - |a| > 1$.

iii. *Another calculation similar to homework and lecture example, but requires some ingenuity and care with details.* [10 marks]

If f and g are conjugate maps, then f and g have the same number of fixed points and the same degree.

Let $\tilde{g}_a(x) = \tilde{f}_a(x) - x = x - \frac{1}{2} + \frac{a}{2\pi}\sin(2\pi x)$. If $\tilde{g}_a(x) = 0$, then $f_a(e^{2\pi i x}) = e^{2\pi i \tilde{f}_a(x)} = e^{2\pi i x}$, so $e^{2\pi i x}$ is a fixed point of f_a. Now, $\tilde{g}(0) = -\frac{1}{2}$, $\tilde{g}(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}\frac{a}{2\pi}\sin(\pi) = 0$, and $\tilde{g}(1) = \frac{1}{2}$, so $e^{2\pi i/2}$ is a fixed point of f. $\tilde{g}_a'(x) = 1 + a\cos(2\pi x)$, so if $a > 1$, $\tilde{g}_a'(\frac{1}{2}) = 1 + a\cos(\pi) = 1 - a < 0$.

$$\tilde{g}_a'(\frac{1}{2}) = \lim_{h \to 0} \frac{\tilde{g}_a(\frac{1}{2} + h) - \tilde{g}_a(\frac{1}{2})}{h} = \lim_{h \to 0} \frac{\tilde{g}_a(\frac{1}{2} + h)}{h} < 0,$$

so for h positive but sufficiently small, $\tilde{g}_a(\frac{1}{2} - h) > 0$. Thus there exists y with $0 < y < \frac{1}{2}$ such that $\tilde{g}_a(y) > 0$. Since $\tilde{g}_a(0) = -\frac{1}{2} < 0$, by the Intermediate Value Theorem, $\tilde{g}_a(x) = 0$ for some $x \in (0, y)$. Then $e^{2\pi i x}$ is a fixed point of f different from $e^{\pi i}$. So f has at least 2 fixed points, so cannot be conjugate to $z \mapsto z^2$. f cannot be conjugate to $z \mapsto z^d$ for $d \neq 2$ since $\deg(f) = 2$. So f is not conjugate to any map $z \mapsto z^d$.

4. (a) *Stating a definition from lecture notes.* [5 marks]

Let f be a monotone map, $\tilde{f}$ a lift of f and $x \in \mathbb{R}$. Define

$$\rho(\tilde{f}, x) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n}.$$

if this limit exists.

The limit $\rho(\tilde{f}, x)$ exists for every $x \in \mathbb{R}$ and is independent of x. Let $\rho(\tilde{f})$ be this value.

(b) i. *Standard calculation.* [2 marks]

$\tilde{f}_a'(x) = 1 + \sin^2(2\pi x)\cos(2\pi x) > 0.$

ii. *Standard calculation.* [3 marks]

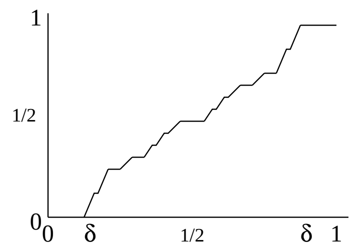
$\tilde{f}_a(x) = x \Leftrightarrow a + \frac{1}{6\pi}\sin^3(2\pi x) = 0 \Leftrightarrow \sin^3(2\pi x) = -6\pi a$. So there is a fixed point iff $-1 \leqslant -6\pi a \leqslant 1$, or $|a| \leqslant \frac{1}{6\pi}$. Thus $\delta = 1/6\pi$.

iii. *Similar to example from lectures and homeworks. Some ingenuity helps.* [4 marks]

If $\tilde{f}_a$ has a fixed point, then $\rho(\tilde{f}_a) = 0$, so $\rho(\tilde{f}_0) = \rho(\tilde{f}_\delta) = \rho(\tilde{f}_{-\delta}) = 0$. $\rho(\tilde{f}_1) = \rho(\tilde{f}_0) + 1 = 1$, and $\rho(\tilde{f}_{1-\delta}) = \rho(\tilde{f}_{-\delta}) + 1 = 1$. $\tilde{f}_{1/2}(0) = \frac{1}{2}$ and $\tilde{f}_{1/2}(\frac{1}{2}) = 1 + \frac{1}{6\pi}\sin^3(\pi) = 1$, so $\tilde{f}^n_{1/2}(0) = n/2$, so

$$\rho(\tilde{f}_{1/2}) = \lim_{n\to\infty} \frac{n/2 - 0}{n} = \frac{1}{2}.$$

iv. *Recalling similar example from lectures.* [2 marks]



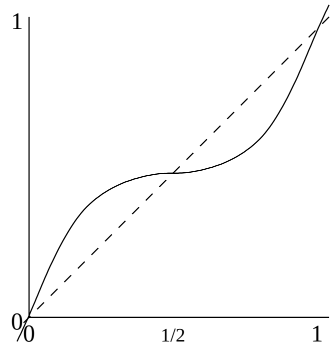
v. *Piecing together standard facts from lectures.* [4 marks]

Since $a \mapsto \rho(f_a)$ is continuous, by the intermediate value theorem, there is a point $a \in (\delta, \frac{1}{2})$ with $\rho(f_a) = 1/3$. Since the maps f_a are increasing with a, the set of values for which $\rho(f_a) = 1/3$ is connected. Thus $\{a : \rho(f_a) = 1/3\}$ is a closed interval or a point in $(\delta, 1/2)$.

5. (a) *Stating a definition from lectures.* [4 marks]

A wandering set is an open set U such that $f^n(U) \cap U = \varnothing$ for all $n \in \mathbb{N}$. A point x is nonwandering if it is not contained in any wandering set.

(b) i. *Simple graph sketch similar to examples from lectures and homeworks.* [2 marks]



ii. *Based on example from homeworks.* [7 marks]

If $0 < x < \frac{1}{2}$, $\tilde{f}_0(x) = x + \frac{1}{2\pi}\sin(2\pi x) > x$, So if $x < y < \tilde{f}_0(x)$, then $0 < x < y < \tilde{f}_0(x) < \tilde{f}_0^n(x) < \tilde{f}_0^n(y) < \tilde{f}_0^{n+1}(x) < \frac{1}{2}$. Therefore $\tilde{f}_0^n(x, \tilde{f}_0(x)) = (\tilde{f}_0^n(x), \tilde{f}_0^{n+1}(x))$ does not intersect $(x, \tilde{f}_0(x))$, so is wandering.

If $\frac{1}{2} < x < 1$, $\tilde{f}_0(x) = x + \frac{1}{2\pi}\sin(2\pi x) < x$, So if $\tilde{f}_0(x) < y < x$, then $\frac{1}{2} < \tilde{f}_0^{n+1}(x) < \tilde{f}_0^n(y) < \tilde{f}_0^n(x) < \tilde{f}_0(x) < y < x < 1$. Therefore $\tilde{f}_0^n(\tilde{f}_0(x), x) = (\tilde{f}_0^{n+1}(x), \tilde{f}_0^n(x))$ does not intersect $(\tilde{f}_0(x), x)$, so is wandering.

iii. *This part requires students to piece together three related results from lectures.* [7 marks]

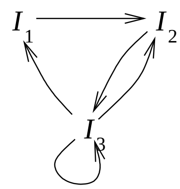
$\tilde{f}_a'(x) = 1 + \cos(2\pi x)$ and $\tilde{f}_a''(x) = -2\pi\sin(2\pi x)$, so $\tilde{f}_a$ is twice-differentiable. Thus f_a is a twice-differentiable monotone degree-one circle map with irrational rotation number. Then by Denjoy's theorem, f_a is conjugate to the rigid rotation $R_{\rho(f_a)}$. Let h be the conjugacy. Since h is a conjugacy, $h(\Omega(f)) = \Omega(R_{\rho(f_a)})$. For any rigid rotation R_α, $\Omega(R_\alpha) = S^1$. Thus $\Omega(f) = h^{-1}(\Omega(R_{\rho(f_a)})) = h^{-1}(S^1) = S^1$.

6. (a) *Standard question based on lecture and homework examples.* [7 marks]

f is linear on each of I_1, I_2, I_3, and $f(0) = 1$, $f(1) = 2$, $f(2) = 3$ and $f(3) = 0$. Thus

$$\begin{aligned} f(I_1) &= f([0,1]) = [1,2] = I_2 \\ f(I_2) &= f([1,2]) = [2,3] = I_3 \\ f(I_3) &= f([2,3]) = [0,3] = I_1 \cup I_2 \cup I_3 \end{aligned}$$

The shift is



The transition matrix is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(b) *Computation similar to lecture and homework computations.* [8 marks]

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{pmatrix}$$

$$A^4 = A^2A^2 = \begin{pmatrix} 1 & & \\ & 3 & \\ & & 7 \end{pmatrix} \; A^5 = A^2A^3 = \begin{pmatrix} 2 & & \\ & 6 & \\ & & 13 \end{pmatrix} \; A^6 = \begin{pmatrix} 4 & & \\ & 11 & \\ & & 24 \end{pmatrix}$$

n	1	2	3	4	5	6
Period n points	1	3	7	11	21	39
Least period n points	1	2	6	8	20	30
Period n orbits	1	1	2	2	4	5

(c) *Standard computation.* [5 marks]

Let x be the period-4 point with itinerary $1233\ldots$. $f(x) = x + 1$, $f^2(x) = (x+1) + 1 = x + 2$, $f^3(x) = 9 - 3(x+2) = 3 - 3x$, $f^4(x) = 9 - 3(3 - 3x) = 9x = x$, so $x = 0$. The orbit of 0 is $(0, 1, 2, 3, 0, \ldots)$.

Let x be the period-4 point with itinerary $2333\ldots$. $f(x) = x + 1$, $f^2(x) = 9 - 3(x+1) = 6 - 3x$, $f^3(x) = 9 - 3(6 - 3x) = 9x - 9$, $f^4(x) = 9 - 3(9x - 9) = 36 - 27x = x$, so $28x = 36$, $x = 9/7$. The orbit of $9/7$ is $(9/7, 16/7, 15/7, 18/7, 9/7\ldots)$.

7. (a) *Stating definition from lectures.* [3 marks]

Let $x_0 < x_1 < \ldots < x_{n-1}$ be the points of the periodic orbit. The pattern of the orbit is the cyclic permutation π such that $f(x_i) = x_{\pi(i)}$.

(b) *Stating definition from lectures.* [3 marks]

A periodic orbit P forces an orbit Q if any orbit with pattern P has orbit with pattern Q.

(c) i. *An example used to illustrate Sharkovskii's theorem* [6 marks]

A pattern is (1 5 2 6 3 4). Let $I_k = [k-1, k]$ for $1 \leqslant k \leqslant 5$. Then $I_1 \mapsto I_5$, $I_2 \mapsto I_4 \cup I_5$, $I_3 \mapsto I_1 \cup I_2 \cup I_3$, $I_4 \mapsto I_1$, $I_5 \mapsto I_2$ There is a fixed point with itinerary $\overline{3}$ and period $2n$ orbits with itineraries $\overline{(2\ 5)^{2(n-1)} 4\ 1}$

ii. *An example similar to homeworks.* [4 marks]

A pattern is (1 2 3 4 5). Let $I_k = [k-1, k]$ for $1 \leqslant k \leqslant 5$. Then $I_1 \mapsto I_2$, $I_2 \mapsto I_3$, $I_3 \mapsto I_4$, $I_4 \mapsto I_5$, $I_5 \mapsto I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$. There is a fixed point with itinerary $\overline{5}$, and period n orbits with itineraries $\overline{5^{n-1} 4}$ for $n \geqslant 2$.

iii. *A question requiring piecing together understanding from lectures and homeworks.* [4 marks]

This is not possible. If P forces a periodic orbit Q of period 5, then Q forces all periods except 3 by Sharkovskii's theorem; in particular, Q forces period 7.

8. (a) *Stating a definition from lectures.* [3 marks]

Let $I_0 = [a,c]$ and $I_1 = [c,b]$. The kneading invariant of f is the itinerary of $f(c)$.

(b) *Giving a procedure from lectures. Two possibilities.* [4 marks]

Suppose the itinerary $s(x) = s_0(x)s_1(x)\ldots$ where $x \in s_i(x)$. Define a sequence $a_i(x)$ by $a_0 = s_0$, $a_i = a_{i-1}$ if $s_i = 0$ and $a_i = 1 - a_{i-1}$ if $s_i = 1$. Then $s(x) < s(y)$ if $a_i(x) = a_i(y)$ for $i < j$ and $a_j(x) < a_j(y)$.

Suppose $s_i(x) = s_i(y)$ for $i < j$ but $s_j(x) \neq s_j(y)$. Then

$$s(x) < s(y) \text{ if } (-1)^{\sum_{i=0}^{j-1} s_i(x)}(s_j(x) - s_j(y)) < 0.$$

(c) *Stating a condition from lectures* [2 marks]

k is the kneading invariant of a unimodal map if $\sigma^n(k) \leqslant k$ for any n, where σ is the shift map.

i. *Standard calculations similar to lecture and homework examples.* [3 marks]

$$\begin{array}{llll} k_1 = 10011010\ldots & a = 1110110\ldots & \sigma^3(k_1) = 101010\ldots & a = 10\ldots \\ \sigma(k_1) = 00110101\ldots & a = 0\ldots & \sigma^4(k_1) = 101010\ldots & a = 110\ldots \\ \sigma^2(k_1) = 01101010\ldots & a = 0\ldots & \sigma^5(k_1) = 010101\ldots & a = 0\ldots \end{array}$$

So this is an acceptable kneading sequence.

ii. *Standard calculation.* [2 marks]

$$\begin{aligned} k_2 &= 1001010011\ldots\ 1110011101\ldots \\ \sigma^5(k_2) &= 1001111111\ldots\ 11101\ldots \end{aligned}$$

So this is not acceptable kneading sequence, as $\sigma^5 k_2 > k_2$.

iii. *Standard calculation.* [2 marks]

$$\begin{array}{llll} k_3 = 10010100\ldots & a = 11100\ldots & \sigma^3(k_3) = 1010010\ldots & a = 110\ldots \\ \sigma(k_3) = 00101001\ldots & a = 0\ldots & \sigma^4(k_3) = 0100101\ldots & a = 0\ldots \\ \sigma^2(k_3) = 01010010\ldots & a = 0\ldots & \sigma^5(k_3) = k & \end{array}$$

So this is an acceptable kneading sequence.

(d) *Standard calculation and application of result from lectures.* [4 marks]

$$\begin{aligned} s(x) &= 100111\ldots\ 111010\ldots \\ s(f(x)) = \sigma(s(x)) &= 001111\ldots\ 001010\ldots \\ s(f^2(x)) = \sigma^2(s(x)) &= 011110\ldots\ 010100\ldots \\ s(f^3(x)) &= 111100\ldots\ 101000\ldots \\ s(f^4(x)) &= 111001\ldots\ 101110\ldots \\ s(f^5(x)) &= 110011\ldots\ 100010\ldots \end{aligned}$$

So $f(x) < f^2(x) < f^5(x) < f^3(x) < f^4(x) < x$. Since $k_3 < s(x) < k_1$, s is the itineraty of a periodic orbit for a map with kneading invariant k_1, but not k_3.