1.

a) Define the *degree* of a circle map f, and justify why your definition makes sense.

b) In each of the following, find a homotopy between f_1 and a suitable f_0 of the form $f_0(z) = \pm z^d$, and hence determine $\deg(f_1)$. In each case, you should justify your choice of homotopy.

(i) $f_1(z) = \dfrac{3z+i}{3-iz}$

(ii) $f_1(z) = \dfrac{z^3 - 4z^2 + 2}{|z^3 - 4z^2 + 2|}$ [20 marks]

2. Let f be a circle map of degree $d > 1$, $\tilde{f}$ a lift of f, $g_1(x) = \tilde{f}(x) - x$ and $g_2(x) = \tilde{f}^2(x) - x$.

(i) Show that if $g_1(x) = k \in \mathbb{Z}$ then $e^{2\pi i x}$ is a fixed point of f. What can you say if $g_2(x) \in \mathbb{Z}$?

(ii) Show that if $g_1(x) = k \in \mathbb{Z}$ then $g_2(x) = k(d+1)$.

[You may assume $\tilde{f}(x+k) = \tilde{f}(x) + k\deg(f)$ for any $k \in \mathbb{Z}$.]

(iii) Compute $g_2(x+1) - g_2(x)$.

(iv) Let x_0 be a point such that $0 < g_2(x_0) < 1$. Sketch a possible graph of $g_2(x)$ for $x_0 \le x \le x_0 + 1$

(v) Find a lower bound for the number of solutions of $g_2(x) \in \mathbb{Z}$ for $x \in [x_0, x_0+1]$ naming any theorem that you use.

[You may want to write x_n for a solution of $g_2(x) = n$.]

(vi) Of your solutions to $g_2(x) = n$, how many can be solutions to $g_1(x) \in \mathbb{Z}$? Find a lower bound for the number of solutions of $g_2(x) \in \mathbb{Z}$ with $x_0 \le x \le x_0 + 1$ which are *not* solutions of $g_1(x) \in \mathbb{Z}$.

(vii) Show that f has at least $d(d-1)$ points of *least* period 2. [20 marks]

3.

a) State a theorem giving conditions under which a circle map f is semiconjugate to the circle map $z \mapsto z^d$ via a degree-one semiconjugacy. State a condition under which there is a conjugacy from f to $z \mapsto z^d$.

b) For $a \in \mathbb{R}$, let $f_a : S^1 \to S^1$ be the circle map with lift

$$\tilde{f}_a(x) = 2x - \tfrac{1}{2} + \tfrac{a}{2\pi} \sin(2\pi x).$$

(i) Give a value of d such that f_a is semiconjugate via a degree-one map to $z \mapsto z^d$.

(ii) Show that if $|a| < 1$, then there is a conjugacy from f_a to $z \mapsto z^d$.

(iii) Suppose $a > 1$. Show that $\tilde{f}_a$ has a fixed point in $(0, 1/2)$. Hence or otherwise deduce that f_a is not conjugate to any map $z \mapsto z^d$, stating clearly any properties of conjugate maps that you use. [20 marks]

4.

a) Define the *rotation number* for a lift $\tilde{f}$ of a degree-one monotone circle map f. State any results which are needed for your definition to make sense.

b) For $a \in \mathbb{R}$, let $f_a : S^1 \to S^1$ be the circle map with lift

$$\tilde{f}_a(x) = x + a + \tfrac{1}{6\pi} \sin^3(2\pi x).$$

(i) Show that $\tilde{f}_a$ is strictly increasing.

(ii) Find a number δ such that $\tilde{f}_a$ has a fixed point if and only if $|a| \leq \delta$.

(iii) Compute $\rho(\tilde{f}_a)$ for $a = 0,\ \delta,\ \tfrac{1}{2},\ 1 - \delta$ and 1.

(iv) Sketch a graph of $\rho(\tilde{f}_a)$ for $0 \leq a \leq 1$.

(v) What can you say about the set of values of a for which $\rho(\tilde{f}_a) = \tfrac{1}{3}$? Justify your answer. [20 marks]

5.

a) Let f be any map. Define a *wandering set* of f and the set of *nonwandering points* of f.

b) Let f_a be the circle map with lift $\tilde{f}_a(x) = x + a + \frac{1}{2\pi}\sin(2\pi x)$, which is strictly increasing.

(i) Sketch the graph of $\tilde{f}_0$.

(ii) Show that if $0 < x < \frac{1}{2}$, the interval $(x, \tilde{f}_0(x))$ is a wandering set of f_0, and if $\frac{1}{2} < x < 1$, the interval $(\tilde{f}_0(x), x)$ is a wandering set of f_0.

(iii) Show that if a is such that $\rho(\tilde{f}_a)$ is irrational, the nonwandering set of f_a is the circle. You should state clearly, but need not prove, any results which you use in your answer. [20 marks]

6. Let $f : [0,3] \to [0,3]$ be given by

$$f(x) = \begin{cases} x+1 & \text{if } 0 \le x \le 2 \\ 9-3x & \text{if } 2 < x \le 3 \end{cases}$$

Let $I_1 = [0,1]$, $I_2 = [1,2]$ and $I_3 = [2,3]$.

(i) Find $f(I_n)$ for $n = 1,2,3$ and hence write down the shift Σ giving the itineraries of f and the transition matrix of this shift.

(ii) Compute the number of periodic orbits of period n for $n = 1, \ldots, 6$.

(iii) Compute the periodic orbits of period 4. [20 marks]

7.

a) Define the *pattern* of a periodic orbit of an interval map.

b) What does it mean for a periodic orbit of an interval map to *force* another?

c) For each of the three parts below, give an example of a pattern satisfying the conditions, or explain why such an orbit does not exist.

(i) A periodic orbit of period 6 which forces all even periods and period 1.

(ii) A periodic orbit of period 5 which forces orbits of all periods.

(iii) A periodic orbit of period 9 which forces all periods except 3 and 7. [20 marks]

8.

a) Define the *kneading invariant* of a unimodal map f of $[a, b]$ with critical point c.

b) Show how to order the itineraries $s^{\pm}(x)$ of points x under f such that if $s^{+}(x) < s^{-}(y)$ if $x < y$.

c) State a condition under which k is the kneading invariant of a unimodal map, and determine which of the following sequences can be a kneading invariant:

(i) $k_1 = 1001\overline{10}$

(ii) $k_2 = 10010100\overline{1}$

(iii) $k_3 = \overline{10010}$

d) Order the points of the periodic orbit with itinerary $\overline{100111}$ on the real line. For the sequences above which are kneading sequences of a unimodal map f, determine whether f has a periodic orbit with the given itinerary. [20 marks]