1 (a) [*Standard problem*] The total number of rearrangements ('anagrams') is $10!/(3!2!) = 302400$ since there are three Es and two Ss. [1 mark]

(i) For no two Es to come together consider the remaining seven letters M R S Y S I D. The three Es can now go in the 8 spaces between these letters and at the ends. This gives $\binom{8}{3}$ choices. There are also $\frac{7!}{2!}$ rearrangements of the seven letters, hence $\binom{8}{3}\frac{7!}{2!} = 141120$ rearrangements in which no two Es come together. [3 marks]

(ii) The number of rearrangements in which at least two come together is, from (i), $302400 - 141120 = 161280$. For the number in which exactly two Es come together we subtract from this the number in which all three Es come together. For the latter we treat the three Es as one letter, giving $8!/2! = 20160$ rearrangements. Subtracting from (i) gives 141120 as the answer. Alternatively, there are 8 places for EE and 7 places for E and $7!/2!$ rearrangements of the other 7 letters, giving $8 \times 7 \times (7!/2!)$ altogether, which gives the same answer, 141120. [3 marks]

(b) [*Standard problem*] The possible totals of the three numbers are from $1+2+3 = 6$ to $38+39+40 = 117$, which is 112 possible totals. We want the smallest value of k for which $\binom{k}{3} > 112$ since this guarantees that there are more than 112 3-element subsets of S and by the pigeonhole principle two of them at least will have the same sum. We find that $\binom{9}{3} = 84$ and $\binom{10}{3} = 120$ so the answer is $k = 10$. [6 marks]

(c) [*Unseen but familiar ideas*] The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. Now $(1+x)^n(1+x)^n = \left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{n}x^n\right)$ and adding up the coefficients of x^n we get

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n}\binom{n}{0}$$

which is equal to

$\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$ since we always have $\binom{n}{k} = \binom{n}{n-k}$. Hence this sum equals $\binom{2n}{n}$.

Now given n men and n women we consider in succession the subsets with $r = 0, 1, 2, \dots, n$ men and the same number of women. For r men and r women we can choose the men in $\binom{n}{r}$ ways and similarly for the women. Thus the number of subsets with r men and r women is $\binom{n}{r}^2$. Adding these up for all values of r (including $r = 0$) and using the above formula gives the total as $\binom{2n}{n}$ subsets. [7 marks]

2 [*Bookwork*] Consider $n + r - 1$ crosses (or other symbols) in a straight line and mark $n - 1$ of them. This makes r unmarked crosses and n spaces between the markers and at the ends (some spaces possibly empty). We can interpret this as n non-negative integers (the number of crosses strictly between markers) adding to r. Hence the number of solutions is precisely $\binom{n+r-1}{n-1}$. [3 marks]

(a)(i) [*Standard problem*] Invent a new variable x_5, also ≥ 0. Then we want the number of solutions in non-negative integers of $x_1 + x_2 + x_3 + x_4 + x_5 = 24$, and this is $\binom{24+5-1}{5-1} = \binom{28}{4} = 20475$. [2 marks]

(ii) If x_1, x_2, x_3, x_4 are all ≥ 3 then replace by $y_i = x_i - 3$ so we are solving $y_1 + y_2 + y_3 + y_4 + x_5 = 12$ with all variables non-negative integers. The number of solutions is $\binom{12+5-1}{5-1} = \binom{16}{4} = 1820$. [3 marks]

(b) [*Standard problem*] First arrange the 12 books in any order. This gives 12! arrangements. Next we want to solve the equation $x_1 + x_2 + x_3 = 12$ with all $x_i > 0$, where x_i is the number of books to be placed on shelf i. Replacing x_i by $y_i + 1$ as above, the number of solutions is $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$, so the total number of arrangements of the 12 books is

$55 \times 12!$ which is in fact 26 345 088 000. [5 marks]

(c) [*Unseen*] Suppose first that the sequence starts with a 1. Let $x_1, x_2, \ldots, x_7$ be the lengths of the blocks, where x_i for i odd refers to the 1s and for i even to the 0s. We are then seeking the number of solutions of

$$x_1 + x_3 + x_5 + x_7 = 8, \quad x_2 + x_4 + x_6 = 7$$

with all the $x_i > 0$. As above replace x_i by $y_i + 1$. The number of solutions to the separate equations are then $\binom{4+4-1}{4-1} = \binom{7}{3} = 35$ and $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$. The total number is the product of these, namely 525.

We must add to this the number of sequences which start with a 0. We are then solving, with all $x_i > 0$,

$$x_1 + x_3 + x_5 + x_7 = 7, \quad x_2 + x_4 + x_6 = 8,$$

the x_i with i odd now referring to numbers of 0s. The number of solutions, by the same method, is

$\binom{3+4-1}{4-1} \times \binom{5+3-1}{3-1} = \binom{6}{3} \times \binom{7}{2} = 20 \times 21 = 420$.

The total number of sequences is therefore $525 + 420 = 945$. [7 marks]

3 [*The theorem and* (a)(i) *are bookwork and the remaining parts of this question are standard problems.*] Hall's Selection Theorem states that if $\{A_i \mid i \in S\}$ is a finite collection of subsets of A then it is possible to choose distinct representatives $x_i \in A_i$ if and only if, for any subset J of S, the union of the corresponding sets A_i has at least $|J|$ elements. [2 marks]

(a) (i) Label the squares w or b as on a chessboard. In any covering a tile will pair a white and a black square. Check that there are the same number of squares of each colour. For each white square w_i write A_i for the set of black squares adjacent to it. A choice of distinct representatives $\{x_i\}$ of the sets A_i will determine a perfect cover, by placing a tile to cover w_i and x_i. Conversely, any perfect cover will determine distinct representatives of the sets A_i in this way. [2 marks]

(ii)

w_1	b_1	■	■	w_2	b_2
■	w_3	b_3	w_4	b_4	■
■	b_5	w_5	b_6	w_6	b_7
b_8	w_7	b_9	w_8	■	w_9
■	■	w_{10}	b_{10}	w_{11}	b_{11}

For (a) the list of sets A_i is as follows, with the unsuccessful attempt to cover the board noted in the bottom line of the table.

w	1	2	3	4	5	6	7	8	9	10	11
A_i	1	2, 4	1, 3, 5	3, 4, 6	3, 5, 6, 9	4, 6, 7	5, 8, 9	6, 9, 10	7, 11	9, 10	10, 11
attempt	1	2	5	3	9	7	8	6	11	10	stuck

[3 marks]

(iii) The algorithm proceeds as follows, where x always refers to white squares and y to black squares. Also x_0 is the first unmatched white square (here 11) and for $i > 0$, y_i is already matched to x_i. The numbers y_i are all distinct. The right-hand column indicates one choice of x which is paired to the chosen y.

$x_0 = 11$	$A_{x_0} = \{10, 11\}$	$y_1 = 10$	x_0
$x_1 = 10$	$A_{x_0} \cup A_{x_1} = \{10, 11, 9\}$	$y_2 = 11$	x_0
$x_2 = 9$	$A_{x_0} \cup A_{x_1} \cup A_{x_2} = \{10, 11, 9, 7\}$	$y_3 = 9$	x_1
$x_3 = 5$	$A_{x_0} \cup A_{x_1} \cup A_{x_2} \cup A_{x_3} = \{10, 11, 9, 7, 3, 5, 6\}$	$y_4 = 7$	x_2
$x_4 = 6$	$A_{x_0} \cup A_{x_1} \cup A_{x_2} \cup A_{x_3} \cup A_{x_4} = \{10, 11, 9, 7, 3, 5, 6, 4\}$	$y_5 = 4$	unmatched

We now proceed to the Endgame:

y_5	x_4	y_4	x_2	y_2	x_0
4	6	7	9	11	11

The matchings $6 \to 7, 9 \to 11$ are replaced by $6 \to 4, 9 \to 7, 11 \to 11$ to give the final matchings

w	1	2	3	4	5	6	7	8	9	10	11
A_i	1	2, 4	1, 3, 5	3, 4, 6	3, 5, 6, 9	4, 6, 7	5, 8, 9	6, 9, 10	7, 11	9, 10	10, 11
final	1	2	5	3	9	4	8	6	7	10	11

[5 marks]

(iv) For (c) the list is

w	1	2	3	4	5	6	7	8	9	10	11
A_i	2, 3	1, 4, 5	2	3, 6	4, 5, 7	6, 9	5, 7, 8, 10	8, 9	6, 9, 11	7, 10	9, 11
forced	3		2	6		9		8	stuck		11

We can prove that no matching exists either by noticing that the choices given in the bottom row are forced, taking in turn $w_3, w_1, w_4, w_6, w_8, w_{11}$ so that there is no choice for w_9; or by noticing that the seven A_i for $i = 1, 3, 4, 6, 9, 11$ have between them only five elements, $2, 3, 6, 9, 11$, violating the Hall condition. [4 marks]

(b) The five sets of presents corresponding to children 2, 4, 5, 8, 9 have only four different presents between them, namely A, D, G, I. Thus a complete matching is impossible. On the other hand the matching

1	2	3	4	5	6	7	8	9
ABCF	ADI	EFGH	DG	DI	ABDEF	BCDEH	ADG	AGI
C	D	F		I	B	E	G	A

successfully matches eight children and presents.

With the extra present J child 2 can have J and child 4 can have D and everyone is happy. [4 marks]

4 [*All the parts of this question are standard, but relatively complicated, calculations.*]

(a) Let A_1, A_2, A_3 be the sets of integers in $\{1, 2, \ldots, 2000\}$ which are divisible by 2, 3, 5 respectively. We need to find $|A_1 \cup A_2 \cup A_3|$. The principle of inclusion-exclusion tells us that this is

$$|A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) + |A_1 \cap A_2 \cap A_3|.$$

Since 2, 3 and 5 are primes we can count these as follows. Note that the number of integers in the given range which are multiples of k is $[2000/k]$.

$$|A_1| = 2000/2 = 1000, \ |A_2| = [2000/3] = 666, \ |A_3| = 2000/5 = 400,$$

$$|A_1 \cap A_2| = [2000/6] = 333, \ |A_2 \cap A_3| = [2000/15] = 133, \ |A_1 \cap A_3| = 2000/10 = 200,$$
$$|A_1 \cap A_2 \cap A_3| = [2000/30] = 66.$$

The required number is $1000 + 666 + 400 - 333 - 133 - 200 + 66 = 1466$. [5 marks]

(b)(i) Given a collection B of squares on a rectangular board, write $r_k(B)$ for the number of ways of selecting k squares of B, no two in the same row or column. The rook polynomial of B is $r(B) = \sum_k r_k(B)x^k$.

Rule 1 If s is any square of B, and B_1 is obtained from B by removing s and B_2 by removing all squares on the same row or column as s, then $r(B) = r(B_1) + xr(B_2)$.

Rule 2 If $B = B_1 \cup B_2$, where no square of B_1 is on the same row or column as a square of B_2, then $r(B) = r(B_1)r(B_2)$. [3 marks]

(ii) In the diagram below s represents the square which is being used in an application of Rule 1.

$$
\begin{aligned}
r\begin{pmatrix} s & & \square \\ \square & \square & \\ & \square & \square \end{pmatrix} &= r\begin{pmatrix} & & \square \\ \square & s & \\ & \square & \square \end{pmatrix} + xr\begin{pmatrix} \square & \\ \square & \square \end{pmatrix} \\
&= r\begin{pmatrix} & & \square \\ \square & & \\ & \square & \square \end{pmatrix} + xr\begin{pmatrix} \square \\ \square \end{pmatrix} + x(1+3x+x^2) \\
&= R(\square)R\begin{pmatrix} & \square \\ \square & \square \end{pmatrix} + x(1+2x) + x(1+3x+x^2) \\
&= (1+x)(1+3x+x^2) + x(1+2x) + x(1+3x+x^2) \\
&= 1+6x+9x^2+2x^3.
\end{aligned}
$$

[5 marks]

(iii) Here we start with a 5×5 board where the five rows represent the five numbers 1, 2, 3, 4, 5. The squares marked $\blacksquare$ are the 'forbidden positions' dictated by the fact that the third row must not have the same entry in the same column as one of the first two rows. We therefore seek the number of ways of placing 5 rooks on the board with the black squares *deleted*.

$$
\begin{matrix}
\blacksquare & & & \blacksquare & \\
& \blacksquare & & & \blacksquare \\
& \blacksquare & \blacksquare & & \\
\blacksquare & & & \blacksquare & \\
& & \blacksquare & & \blacksquare
\end{matrix}
$$

First we calculate the rook polynomial of the 5×5 board B with the black squares the only ones *present*. Note that the first and fourth columns and the first and fourth rows constitute a board B_1 as in Rule 2 which has no square in the same row or column as the rest of the board, B_2. Hence $r(B)$ is the product of $r(B_1)$ and $R(B_2)$. Furthermore B_2 is the board considered in (ii) above. So we have immediately

$$r(B) = (1+4x+2x^2)(1+6x+9x^2+2x^3) = 1+10x+35x^2+50x^3+26x^4+4x^5.$$

The number of ways of selecting m squares no two in the same row or column on an $m \times n$ board ($m \leq n$) with the squares of B forbidden is

$$\frac{1}{(n-m)!}(n! - (n-1)!r_1(B) + (n-2)!r_2(B) - \ldots + (-1)^m(n-m)!r_m(B)).$$

In the present case we have $m = n = 5$ and this gives the number of possibilities for the third row of the Latin square

$$5! - 4! \times 10 + 3! \times 35 - 2! \times 50 + 1! \times 26 - 0! \times 4 = 12.$$

[7 marks]

5 [*All parts except* (v) *are standard. Part* (v) *is unseen but the general formula for* $1/(1-x)^k$ *will be discussed explicitly.*]

(i) Write $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then $(1-2x-3x^2)A(x) = a_0 + (a_1 - 2a_0)x = 2-2x$. Hence

$$A(x) = \frac{2-2x}{1-2x-3x^2} = \frac{2-2x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}.$$

Partial fractions gives $A = 1, B = 1$, and $A(x) = \sum(-x)^n + \sum(3x)^n$, so $a_n = (-1)^n + 3^n$. [3 marks]

(ii)

$$(1-2x-3x^2)A(x) = a_0 + (a_1 - 2a_0)x + (x^2 + x^3 + x^4 + \ldots) = 2 - 2x + \frac{x^2}{1-x}.$$

This gives

$$A(x) = \frac{2-4x+3x^2}{(1-x)(1+x)(1-3x)} = -\frac{\frac{1}{4}}{1-x} + \frac{\frac{9}{8}}{1+x} + \frac{\frac{9}{8}}{1-3x},$$

using partial fractions.

Hence, reading off the coefficient of x^n, $a_n = -\frac{1}{4} + \frac{9}{8}(-1)^n + \frac{9}{8}3^n$. [4 marks]

(iii) Here,

$$(1-4x)A(x) = a_0 + (a_1 - 4a_0)x + (a_2 - 4a_1)x^2 + \ldots = 3 + (2x + (2x)^2 + (2x)^3 + \ldots)$$

$$= 3 + 2x(1 + 2x + (2x)^2 + \ldots) = 3 + \frac{2x}{1-2x} = \frac{3-4x}{1-2x}.$$

Using partial fractions,

$$A(x) = \frac{3-4x}{(1-4x)(1-2x)} = \frac{4}{1-4x} - \frac{1}{1-2x},$$

and reading off the coefficient of x^n, $a_n = 4 \cdot 4^n - 2^n$, which can also be written as $2^n(2^{n+2} - 1)$. [3 marks]

(iv) Here,

$$(1-x-2x^2)A(x) = a_0 + (a_1 - a_0)x + (a_2 - a_1 - 2a_0)x^2 + \ldots = 1 + x^2 - x^3 + x^4 - x^5 + \ldots$$

$$= x + \frac{1}{1+x} = \frac{1+x+x^2}{1+x}.$$

Using partial fractions,

$$A(x) = \frac{1+x+x^2}{(1+x)^2(1-2x)} = -\frac{\frac{1}{9}}{1+x} + \frac{\frac{1}{3}}{(1+x)^2} + \frac{\frac{7}{9}}{1-2x},$$

and reading off the coefficient of x^n gives

$$a_n = -\tfrac{1}{9}(-1)^n + \tfrac{1}{3}(n+1)(-1)^n + \tfrac{7}{9}2^n = \tfrac{2}{9}(-1)^n + \tfrac{1}{3}n(-1)^n + \tfrac{7}{9}2^n.$$

[5 marks]

(v) Here,

$$(1-2x)A(x) = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \ldots = 1 + 0x + 1 \cdot 2x^2 + 2 \cdot 3x^3 + \ldots.$$

Now

$$\frac{1}{(1-x)^3} = 1 + \tfrac{2\cdot 3}{2}x + \tfrac{3\cdot 4}{2}x^2 + \ldots, \text{so } \frac{2x^2}{(1-x)^3} = 1 \cdot 2x^2 + 2 \cdot 3x^3 + 3 \cdot 4x^4 + \ldots$$

and this gives

$$A(x)(1-2x) = 1 + \frac{2x^2}{(1-x)^3} = \frac{1-3x+5x^2-x^3}{(1-x)^3}, \text{so } A(x) = \frac{1-3x+5x^2-x^3}{(1-x)^3(1-2x)}.$$

[5 marks]

[*Not part of question*: using partial fractions we find

$$A(x) = -\frac{2}{1-x} - \frac{2}{(1-x)^3} + \frac{5}{1-2x},$$

which gives $a_n = 5 \cdot 2^n - (n^2 + 3n + 4)$.]

6 [*All parts of this question are bookwork apart from* (iv) *which is unseen.*]

(i) $c_0 = 1,\ c_1 = 1,\ c_2 = c_0c_1 + c_1c_0 = 2,\ c_3 = c_0c_2 + c_1c_1 + c_2c_0 = 5,\ c_4 = 14,\ c_5 = 42$. [2 marks]

(ii) The constant term is 1 on both sides. For $n \geq 0$ the coefficient of x^{n+1} in RHS is the coefficient of x^n in $C(x)^2$ which is $\sum_{r=0}^{n} c_r c_{n-r} = c_{n+1}$. This is the same as the coefficient on LHS. [4 marks]

(iii) Let a_n be the number of shortest paths lying on or above the diagonal, and let b_n be the number which touch the diagonal only at $(0,0)$ and (n,n). We define $a_0 = 1$. Then

(a) $b_n = a_{n-1}$ for all $n > 1$. This is because every shortest path must start by $(0,0) \to (0,1)$ and end by $(n-1,n) \to (n,n)$, so the paths touching the diagonal only at the ends are exactly those on or above the line $y = x + 1$, hence are in one-to-one correspondence with all shortest paths for an $n-1 \times n-1$ grid.

(b) Suppose the shortest path touches the diagonal $x = y$ for the *last time* at (r,r), where $0 \leq r \leq n-1$, apart from ending at (n,n). Considering the part of this path up to (r,r) and beyond (r,r) it follows that the number of such paths is $a_r b_{n-r} = a_r a_{n-r-1}$ by (a). Thus

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-1} a_0, \quad n \geq 1,$$

and clearly $a_1 = 1$. Hence the recurrence and the value a_1 (or indeed a_0) are the same as for the Catalan numbers; hence the two sequences coincide. [4 marks]

(iv) In (iii) think of $+1$ as giving a one-unit step in the y-direction and -1 as giving a one-unit step in the x-direction. The condition on partial sums being ≥ 0 then says precisely that $y-x \geq 0$ at each stage, so that the shortest paths on or above the diagonal $x = y$ in (iii) are in one-to-one correspondence with the sequences having non-negative partial sums. Hence the number is the same, namely c_n. [5 marks]

(v) The formula for the quadratic $xC(x)^2 - C(x) + 1 = 0$ gives $C(x) = (1 \pm \sqrt{1-4x})/2x$ and so $xC(x) = \frac{1}{2}(1 \pm \sqrt{1-4x})$. RHS must give 0 when $x = 0$, so $xC(x) = \frac{1}{2}(1 - (1-4x)^{1/2})$.

Expanding $(1-4x)^{1/2}$ by the binomial theorem gives a sum where the coefficient of x^{n+1} is

$$\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\cdots\frac{1-2n}{2}\frac{(-4)^{n+1}}{(n+1)!}.$$

Cancelling the factors -2 leads to

$$-1.3.5.\ldots.(2n-1)\frac{2^{n+1}}{(n+1)!} = -2\frac{(2n)!}{n!(n+1)!}.$$

Divide by $-2x$ to obtain $C(x)$, and we obtain

$$c_n = \frac{(2n)!}{n!(n+1)!}, \text{ while the given formula is } \frac{1}{n+1}\frac{(2n)!}{n!n!},$$

which is clearly the same. [5 marks]

7 [*This is all bookwork or similar to class or homework questions.*]

Let s_n be the number of solutions of $n = a + 2b + 4c$ in non-negative integers. Then

$$\begin{aligned} S(t) = \sum_{n=1}^{\infty} s_n t^n &= (1 + t + t^2 + \ldots)(1 + t^2 + t^4 + \ldots)(1 + t^4 + t^8 + \ldots) \\ &= 1/(1-t)(1-t^2)(1-t^4). \end{aligned}$$

[3 marks]

Working up to terms in t^9 we have $1/(1-t)(1-t^2) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 + \ldots$ and

$$\begin{aligned} S(t) &= 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 \\ &\quad + t^4(1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5) + t^8(1+t) + \ldots \\ &= 1 + t + 2t^2 + 2t^3 + 4t^4 + 4t^5 + 6t^6 + 6t^7 + 9t^8 + 9t^9 + \ldots. \end{aligned}$$

[3 marks]

The product $\prod_{i=1}^{\infty}(1-t^i)^{-1}$ is a sum of terms $t^{a_1}t^{2a_2}t^{3a_3}\cdots$; by associating this term to the partition with a_i parts equal to i for each i, we see that the terms which multiply out to t^n correspond exactly to the partitions of n, so this product is equal to $P(t)$.

We may argue similarly for partitions into parts of length at most m: we restrict the product to $i \leq m$, so that $P_m(t) = \prod_{i=1}^{m}(1-t^i)^{-1}$. [3 marks]

The Ferrers graph of a partition λ is obtained by arranging the parts of λ in descending order as $\lambda_1, \ldots, \lambda_k$ and then taking a row of λ_1 dots; below the first λ_2 of these a second row of dots, and so on.

The graph of a partition with all parts of length at most m has at most m columns. Interchanging rows and columns carries one Ferrers graph to another. It determines a bijection, which carries a graph with at most m columns to one with at most m rows, representing a partition with at most m parts. Hence the number of partitions of n with at most m parts is equal to the number of partitions of m with all parts at most m. [3 marks]

The generating function $R(t)$ for partitions with at most 4 parts is then the same as $P_4(t)$, namely $R(t) = \prod_{i=1}^{4}(1-t^i)^{-1} = (1-t^3)^{-1}S(t)$. So $R(t) = (1+t^3+t^6+t^9)S(t)$, up to terms in t^9. Thus $R(t) = 1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 + 18t^9 + \ldots$. [2 marks]

The corresponding generating function for partitions with at most 3 parts is $P_3(t)$. Now $P_3(t)(1-t^4)^{-1} = P_4(t) = R(t)$ so $P_3(t) = (1-t^4)R(t) = 1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + 8t^7 + 10t^8 + 12t^9 + \ldots$. [3 marks]

Partitions with exactly 4 parts will be counted by $P_4(t) - P_3(t)$ and hence by $t^4 + t^5 + 2t^6 + 3t^7 + 5t^8 + 6t^9$ up to $n = 9$.

There are thus **6** partitions of 9 with exactly 4 parts. These are (6,1,1,1), (5,2,1,1), (4,3,1,1), (4,2,2,1), (3,3,2,1), (3,2,2,2), with Ferrers graphs

× × × × × ×	× × × × ×	× × × ×	× × × ×	× × ×	× × ×
×	× ×	× × ×	× ×	× × ×	× ×
× ,	× ,	× ,	× × ,	× × ,	× ×
×	×	×	×	×	× ×

[3 marks]

8 [*This is all bookwork or similar to class or homework questions.*]

A function of N variables $\{x_1, \ldots, x_N\}$ is called *symmetric* if it is unchanged by any permutation of the x_i. [This is equivalent to saying it it is unchanged by any transposition of two of the x_i.] The *elementary symmetric function* σ_n is the sum of all products $x_{i_1} \ldots x_{i_n}$ with $1 \le i_1 < \ldots < i_n \le N$. The *power sum symmetric function* π_n is the sum $\sum_{i=1}^{N} x_i^n$. Let us also write $\sigma_0 = 1$, $\pi_0 = N$.

The Newton Identities state that

$$n\sigma_n = \sigma_{n-1}\pi_1 - \sigma_{n-2}\pi_2 + \ldots + (-1)^{n-1}\sigma_0\pi_n$$

for $n \ge 1$.

To prove them, write $E(t) = \sum_0^N \sigma_r t^r$ and $P(t) = \sum_1^\infty \pi_r t^r$. Then $E(t) = \prod_1^N (1 + x_i t)$ so that

$$\ln E(t) = \sum_1^N \ln(1 + x_i t), \text{so that } \frac{E'(t)}{E(t)} = \sum_1^N \frac{x_i}{1 + x_i t}.$$

Further,

$$\begin{aligned} P(t) &= (x_1 + x_2 + \ldots + x_N)t + (x_1^2 + x_2^2 + \ldots + x_N^2)t^2 + (x_1^3 + x_2^3 + \ldots + x_N^3)t^3 + \ldots \\ &= (x_1 t + x_1^2 t^2 + x_1^3 t^3 + \ldots) + (x_2 t + x_2^2 t^2 + x_2^3 t^3 + \ldots) + \ldots + (x_N t + x_N^2 t^2 + x_N^3 t^3 + \ldots) \\ &= \frac{x_1 t}{1 - x_1 t} + \frac{x_2 t}{1 - x_2 t} + \ldots + \frac{x_N t}{1 - x_N t}, \end{aligned}$$

so that

$$\frac{tE'(t)}{E(t)} = -P(-t), \text{ that is } tE'(t) + E(t)P(-t) = 0.$$

The Newton indentities now follow by comparing the coefficients of t^n on the two sides of this equation.

[6 marks]

From the first three Newton identities $\pi_1 = \sigma_1$, $\pi_2 - \pi_1\sigma_1 = -2\sigma_2$, $\pi_3 - \pi_2\sigma_1 + \pi_1\sigma_2 = 3\sigma_3$, we deduce in turn $\pi_2 = \sigma_1^2 - 2\sigma_2$, $\pi_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$. [2 marks]

(i) $1/\alpha + 1/\beta + 1/\gamma = (\beta\gamma + \gamma\alpha + \alpha\beta)/\alpha\beta\gamma = \sigma_2/\sigma_3$.

$1/\alpha\beta + 1/\beta\gamma + 1/\gamma\alpha = (\alpha + \beta + \gamma)/\alpha\beta\gamma = \sigma_1/\sigma_3$

$1/\alpha\beta\gamma = 1/\sigma_3$. [2 marks]

(ii)

$$\begin{aligned} s_1 &= \alpha^2 + \beta^2 + \gamma^2 = \sigma_1^2 - 2\sigma_2, \\ s_2 &= \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = \sigma_3^2(1/\alpha^2 + 1/\beta^2 + 1/\gamma^2) \\ &= \sigma_3^2((\sigma_2/\sigma_3)^2 - 2\sigma_1/\sigma_3) = \sigma_2^2 - 2\sigma_1\sigma_3, \\ s_3 &= \alpha^2\beta^2\gamma^2 = \sigma_3^2. \end{aligned}$$

The required equation is $x^3 - s_1 x^2 + s_2 x - s_3 = 0$. [4 marks]

Each determinant δ_r changes sign when two columns are interchanged, hence if any two of α, β and γ are interchanged the quotient ϕ_r is unchanged, and thus is a symmetric function. [2 marks]

Manipulating determinants via subtracting one column from another and taking out common factors leads to expressing δ_2 as $(\alpha - \beta)(\gamma - \alpha)(\beta - \gamma)$ and δ_4 as this multiplied by

$$\alpha^2 + \alpha\beta + \beta^2 + \alpha\gamma + \beta\gamma + \gamma^2 = \sigma_1^2 - \sigma_2.$$

[4 marks]