THE UNIVERSITY
of LIVERPOOL

1. (a) Write down the number of rearrangements of the letters in the word MERSEYSIDE. Calculate the number of these in which

(i) no two Es come together;

(ii) exactly two Es come together.

(b) Let X be the set $\{1, 2, 3, 4, \ldots, 40\}$. A subset S is chosen from X, with $|S| = k$, say. Find the smallest value of k which guarantees that two subsets of S each with three members have the same sum. [The two subsets of S are allowed to overlap but not of course to be identical.]

(c) Write down the coefficient of x^n in the binomial expansion of $(1+x)^{2n}$ and by writing this also as $(1+x)^n(1+x)^n$, or otherwise, deduce that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Given n men and n women, show that the number of subsets of the $2n$ people which contain the same number of men and women is $\binom{2n}{n}$. [Note that this includes the empty set, with no men and no women!]

2.

Explain how to calculate the number of integer solutions of the equation

$$x_1 + x_2 + x_3 + \ldots + x_n = r,$$

where all the x_i are ≥ 0.

(a) Find the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 < 25,$$

where the x_i are (i) all ≥ 0, (ii) all ≥ 3.

(b) I have 12 books, all different. How many different ways are there of placing the books on 3 shelves, with at least one book on each shelf? [Note that the order of the books left to right on any shelf is important.]

(c) Consider a sequence of 15 symbols, each a 0 or a 1, in which there are eight 1's and seven 0's. For example

111001011100001

This sequence has seven 'blocks' of equal symbols, namely 111, 00, 1, 0, 111, 0000, 1. Calculate the number of sequences of 15 symbols 0, 1 with eight 1's and seven 0's which contain seven blocks. Note that the sequence can start with either a 1 or a 0.

3. State Hall's selection theorem.

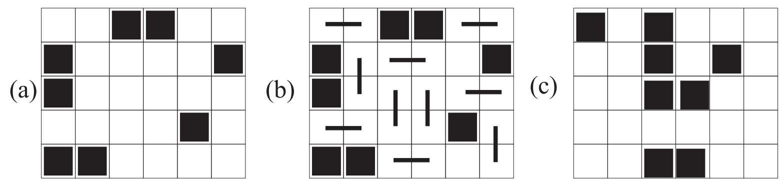
(a) A rectangular array of squares is given, from which a subset B of the squares is excluded.

(i) Show how the problem of constructing a perfect cover of the remaining squares by 2×1 tiles can be reformulated in terms of selecting distinct representatives for a suitable collection of sets, describing carefully what sets are to be used.

(ii) For the array (a) below, where excluded squares are shown as ■, list a suitable collection of sets for this reformulation.

(iii) Starting from the unsuccessful attempt (b) to cover the unmarked squares by 2×1 tiles, *use the algorithm described in class* to find a perfect cover.

(iv) For the board (c) either find a perfect cover by 2×1 tiles or prove that none exists.



(b)

Nine children $1, 2, \ldots, 9$ are invited to a children's party organized by Mr Hall, whose own children are 1, 2 and 3. Nine different presents A, B, ..., I have been bought for the children and after much consultation with all the parents Mr Hall draws up a table showing which presents are acceptable for the various children:

1	2	3	4	5	6	7	8	9
ABCF	ADI	EFGH	DG	DI	ABDEF	BCDEH	ADG	AGI

Prove that eight, but not all nine, of the children can be given a suitable present.

A tenth present, J, is purchased which Mr Hall knows will be suitable for any of his own children. Does this help the situation? [Justify your answers clearly.]

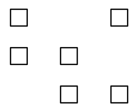
[20 marks]

THE UNIVERSITY *of* LIVERPOOL

4. (a) Use the principle of inclusion-exclusion to find the number of integers in $\{1, 2, 3, \ldots, 2000\}$ which are divisible by at least one of 2, 3 and 5. (State the form of the principle which you are using.)

(b) (i) Define a *rook polynomial*. Give rules which will enable the rook polynomial of any board to be calculated.

(ii) Calculate the rook polynomial of the 3×3 board shown. The open squares represent the six squares actually present and the blank spaces the three squares pruned.



(iii) The first two rows of a Latin square are

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{matrix}$$

Use the forbidden positions formula (which should be clearly stated) to find how many possibilities there are for the third row. [20 marks]

5. Solve the following recurrence relations, (i)–(iv). In each case you should find an expression for a_n and for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 2a_{n+1} + 3a_n$, $a_0 = a_1 = 2$.

(ii) $a_{n+2} = 2a_{n+1} + 3a_n + 1$, $a_0 = a_1 = 2$.

(iii) $a_{n+1} = 4a_n + 2^{n+1}$, $a_0 = 3$.

(iv) $a_{n+2} = a_{n+1} + 2a_n + (-1)^n$, $a_0 = a_1 = 1$.

(v) For the following recurrence relation, find the generating function $A(x)$ only.

$$a_{n+1} = 2a_n + n(n+1), \quad a_0 = 1.$$

[20 marks]

6. The Catalan numbers $\{c_n\}$ are defined recursively by the relations

$$c_0 = 1, c_{n+1} = \sum_{r=0}^{n} c_r c_{n-r}, \; n \geq 0.$$

(i) Write down the sequence of these numbers up to c_5.

(ii) Prove, by comparing the terms in x^{n+1}, that the generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

for the Catalan numbers satisfies the quadratic equation

$$C(x) = 1 + xC(x)^2.$$

(iii) Write a_n for the number of shortest paths from $(0,0)$ to (n,n) in the (x,y)-plane, every segment of unit length being along one of the grid lines $x = r$ or $y = s$ where r and s are integers in $\{0, 1, \ldots, n\}$, these paths never going below the diagonal $x = y$. Find a recurrence relation for a_n and deduce that $a_n = c_n$ for all $n \geq 1$.

(iv) Explain why c_n also counts the number of sequences containing n numbers 1 and n numbers -1 such that every partial sum of the sequence is ≥ 0. [For example, with $n = 3$, the sequence $1, 1, -1, 1, -1, -1$ has this property since the partial sums are $1, 1+1, 1+1-1, 1+1-1+1, 1+1-1+1-1$ and $1+1-1+1-1-1$, all of which are ≥ 0. It may help to consider an analogy with (iii).]

(v) Using the equation $C(x) = 1 + xC(x)^2$ show that

$$xC(x) = \frac{1}{2}(1 - \sqrt{1-4x}),$$

and deduce that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

[20 marks]

THE UNIVERSITY *of* LIVERPOOL

7. Obtain a formula for the generating function $S(t) = \sum_{n=1}^{\infty} s_n t^n$, where s_n is the number of solutions of $n = a + 2b + 4c$ in non-negative integers.

Calculate $S(t)$ up to the term in t^9.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer n. Find also the function $P_m(t)$ which enumerates the number of partitions of n into parts of length $\leq m$.

Define the term *Ferrers graph*. Use Ferrers graphs to establish a bijection between partitions of n with at most m parts and partitions of n into parts of length $\leq m$.

Hence write down the generating function $R(t)$ which enumerates partitions of n with at most 4 parts.

Calculate $R(t)$ up to the term in t^9, and hence find the number of partitions of 9 with at most 4 parts.

Show that $(1 - t^4)R(t)$ enumerates partitions of n with at most 3 parts.

Use your calculations to determine the number of partitions of n with exactly 4 parts for all $n \leq 9$. Exhibit the corresponding Ferrers graphs in the case $n = 9$.

[20 marks]

8. Define the term *symmetric function*. For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

Express π_3 in terms of the elementary symmetric functions.

Obtain, in terms of the elementary symmetric functions of α, β and γ,

(i) the elementary symmetric functions of $1/\alpha, 1/\beta, 1/\gamma$, and

(ii) the equation with roots $\alpha^2,\ \beta^2,\ \gamma^2$.

Write δ_r for the determinant

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^r & \beta^r & \gamma^r \end{pmatrix}$$

Show that, for each $r \geq 3$, $\phi_r = \delta_r/\delta_2$ is a symmetric function of α, β and γ. Express ϕ_4 in terms of their elementary symmetric functions.

[20 marks]