$\boxed{1}$ (a) $(x+y)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r$. Putting $x = 1$ and $y = -1$ gives $0 = \sum_{r=0}^n (-1)^r \binom{n}{r}$.

With $x = 1, y = 1$ we have $2^n = \sum_{r=0}^n \binom{n}{r}$.

Add, and divide by 2 to get $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots$; the odd terms are equal to the even terms by the first equation.

(b) $\binom{2n}{2}$ counts the number of ways of selecting two objects from $2n$. Make the selection by grouping the $2n$ objects into two groups of n. Then either select 2 from the first group, or the second group each in $\binom{n}{2}$ ways, or select one from each group, by choosing one from the first group in n ways followed by one from the second group, also in n ways, making $2\binom{n}{2}$ for the selections of pairs in the same group, plus n^2 for the choices from the separate groups.

Group the kn objects into k groups of n. Count selections of pairs from single groups in $k\binom{n}{2}$ ways and pairs from separate groups in $\binom{k}{2} \times n^2$ ways, by first selecting the pair of groups, and then the elements one from each, making a total of $k\binom{n}{2} + \binom{k}{2} \times n^2$.

(c) There are $\dfrac{11!}{6!2!} = \mathbf{27720}$ anagrams.

For each anagram of the 5 letters $ALLMW$ insert the six Os in the six positions $\times$ between the letters or at either end $(\times A \times L \times L \times M \times W \times)$. There are $\dfrac{5!}{2!} = 60$ anagrams and one way to place the letters O, making **60** ways in total.

For each anagram of the 5 letters $ALLMW$ insert OO in any of the six positions between the letters or at either end. Then insert the remaining four Os in any of the five positions $\times$ which are left. There are $\dfrac{5!}{2!} = 60$ anagrams, 6 ways to place the OO and $\binom{5}{4} = 5$ ways to place the four single letters O, making **1800** ways in total.

(d) (i) All three numbers in the subset must be odd. Since there are 6 odd numbers available there are $\binom{6}{3} = \mathbf{20}$ choices with all three odd.

(ii) Either all three numbers in the subset are odd, making $\binom{6}{3} = 20$ subsets, or two are even and one is odd. Choose the odd number in 6 ways and the two even numbers in $\binom{7}{2} = 21$ ways, making a further $6 \times 21 = 126$ subsets. The total number of possibilities is then $20 + 126 = \mathbf{146}$ subsets.

(iii) There are altogether $\binom{6}{3} + \binom{6}{2} + \binom{6}{1} = 20 + 15 + 6 = 41$ non-empty subsets of A with at most 3 numbers. Possible totals range from $-6-5-4 = -15$ to $4+5+6 = 15$, making 31 possibilities, so there are more sets than totals. The pigeonhole principle then gives at least two sets with the same total.

$\boxed{\mathbf{2}}$

(a) (i) We can arrange r objects and $n-1$ boundary markers in a row in $\binom{r+n-1}{n-1}$ ways; now assign the objects between markers $i-1$ and i to container i.

Any solution determines a distribution by putting k_i objects into container i. Conversely, given any distribution, count the number of objects k_i in container i to get a solution to the equations. Write k_i for the number of items of variety i chosen. Then again any choice gives a solution to the equations, while any solution gives the number of items of each variety to be included in the selection, totalling r.

(ii) Write the numbers as $x_1x_2x_3x_4$. Then $x_1+x_2+x_3+x_4=9$ and $x_1 \geq 1, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$. Put $x_1 = 1 + x_1'$ and then count solutions of $x_1' + x_2 + x_3 + x_4 = 8$ in non-negative integers, to get $\binom{8+3}{3} = \mathbf{165}$ solutions.

If no digits are to be 0 then write $x_i = 1 + x_i'$ in each case and solve $x_1' + x_2' + x_3' + x_4' = 5$ in non-negative integers in $\binom{5+3}{3} = \mathbf{56}$ ways.

(b) (i) The resulting codes are all anagrams of the 12 letter word with 8 asterisks and four different numbers, making a total of $\frac{12!}{8!} = 990 \times 12 =$ **11880**. [Or place the numbers 1,2,3,4 in one of 12 different positions].

(ii) First arrange the numbers in order in 4^4 ways. Then put the asterisks in the five places around them, with x_1 asterisks before the first number, then x_2 between the first two numbers, leading to $x_1 + x_2 + \cdots + x_5 = 8$ with $x_i \geq 0$. There are then $\binom{8+4}{4} = 495$ ways to do this making a total of $495 \times 256 = \mathbf{126720}$ codes.

(iii) Arrange the four numbers in order, in 4! ways and then fill in asterisks as before, but this time we must have $x_2, x_3, x_4 > 0$. Take $x_2 = 1 + x_2', x_3 = 1 + x_3', x_4 = 1 + x_4'$ leading to $\binom{5+4}{4}$ arrangements of asterisks, and a total of $4! \times \binom{5+4}{4} = 9 \times 8 \times 7 \times 6 = 9 \times 336 = \mathbf{3024}$ codes.

Alternatively select 4 of the 9 gaps between and at either end of the 8 asterisks, in $\binom{9}{4}$ ways, and then arrange the four numbers in these four places in 4! ways, making a total of $\binom{9}{4} \times 4! = 9 \times 8 \times 7 \times 6 = \mathbf{3024}$ codes as above.

$\boxed{\mathbf{3}}$ Hall's Selection Theorem states that if $\{S_i \mid i \in I\}$ is a finite collection of subsets of S then it is possible to choose distinct representatives $x_i \in S_i$ if and only if, for any subset J of I, the union of the corresponding sets S_i has at least $|J|$ elements.

(a) (i) Label the squares w or b as on a chessboard. In any covering a tile will pair a white and a black square. Check that there are the same number of squares of each colour. For each white square w_i write S_i for the set of black squares adjacent to it. A choice of distinct representatives $\{x_i\}$ of the sets S_i will determine a perfect cover, by placing a tile to cover w_i and x_i. Conversely, any perfect cover will determine distinct representatives of the sets S_i in this way.

(ii)

$$\begin{array}{cccccc} w_1 & b_1 & w_2 & b_2 & w_3 & b_3 \\ b_4 & w_4 & b_5 & w_5 & \blacksquare & \blacksquare \\ w_6 & \blacksquare & \blacksquare & b_6 & w_7 & b_7 \\ \blacksquare & \blacksquare & b_8 & w_8 & \blacksquare & w_9 \\ \blacksquare & b_9 & w_{10} & b_{10} & w_{11} & b_{11} \end{array} \qquad \begin{array}{cccccc} w_1 & b_1 & \blacksquare & \blacksquare & w_2 & \blacksquare \\ b_2 & w_3 & b_3 & \blacksquare & b_4 & w_4 \\ w_5 & b_5 & \blacksquare & \blacksquare & w_6 & b_6 \\ b_7 & \blacksquare & b_8 & w_7 & b_9 & w_8 \\ w_9 & b_{10} & w_{10} & \blacksquare & w_{11} & b_{11} \end{array}$$

For the first board the list is

w	1	2	3	4	5	6	7	8	9	10	11
b	$\mathbf{1}, 4$	$1, \mathbf{2}, 5$	$2, \mathbf{3}$	$1, 4, \mathbf{5}$	$2, 5, \mathbf{6}$	$\mathbf{4}$	$6, \mathbf{7}$	$6, \mathbf{8}, 10$	$7, \mathbf{11}$	$8, \mathbf{9}, 10$	$\mathbf{10}, 11$

For the second board the list is

w	1	2	3	4	5	6	7	8	9	10	11
b	$1, 2$	4	$1, 2, 3, 5$	$4, 6$	$2, 5, 7$	$4, 6, 9$	$8, 9$	$6, 9, 11$	$7, 10$	$8, 10$	$9, 11$

(iii) There is a tiling, indicated in bold on the list for the first board; a tiling could equally be shown instead on a diagram.

For the second board there are 5 white squares $w_2, w_4, w_6, w_8, w_{11}$ with only 4 adjacent black squares b_4, b_6, b_9, b_{11}, so no selection, and hence no tiling, is possible.

(b) (i) One possibility is to house $A, B, C, D, E, F, G, H, I$ in rooms $3, 4, 9, 8, 10, 7, 5, 6, 1$ respectively.

It is not possible to find suitable rooms for all 10 people since only the five rooms $\{1, 4, 7, 9, 10\}$ are suitable for the six people $\{B, C, E, F, I, J\}$.

(ii) In the present room allocation the rooms $1, 9$ and 10 are occupied by I, C, E already. But C is willing to share room 4 with B, leaving room 9 for J.

To accommodate the driver, find who is in rooms 8 and 10.

These are D and E. Moving D to the vacant room 2 allows a suitable allocation.

For example $A, B, C, D, E, F, G, H, I, J$ and the driver in rooms $3, 4, 4, 2, 10, 7, 5, 6, 1, 9, 8$ respectively.

[Many other allocations are possible].

$\boxed{4}$ (i) Given a collection A of squares on a rectangular board, write $r_k(A)$ for the number of ways of selecting k squares of A, no two in the same row or column. The rook polynomial of A is $R(A) = \sum_k r_k(A)x^k$.

If S is any square of A, and A' is obtained from A by removing S and A'' by removing all squares on the same row or column as S, then $R(A) = R(A') + xR(A'')$.

If $A = A_1 \cup A_2$, where no element of A_1 is on the same row or column as an element of A_2, then $R(A) = R(A_1)R(A_2)$.

The number of ways of selecting m squares no two in the same row or column on an $m \times n$ board with the squares of B forbidden is

$$\frac{1}{(n-m)!}(n! - (n-1)!r_1(B) + (n-2)!r_2(B) - \ldots + (-1)^m(n-m)!r_m(B)).$$

(ii) The inclusion-exclusion formula says that if $\{A_i \mid 1 \leq i \leq n\}$ are subsets of U, then the number of elements of U belonging to none of the sets A_i is

$$\sum_{I \subset \{1,\cdots,n\}} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|,$$

where $\bigcap_{i \in \emptyset}$ is interpreted as U.

Apply the formula where U is the set of ways to choose m squares with no two in the same row or column, and the subset A_i are those selections where the square in row i lies in the forbidden set B.

(iii) The rook polynomial can be calculated starting with the marked square as S and applying one of the rules

$$\begin{aligned}
R\begin{pmatrix} \blacksquare & \square & \\ \blacksquare & & \blacksquare \\ & \blacksquare & \blacksquare \end{pmatrix} &= R\begin{pmatrix} \blacksquare & & \\ \blacksquare & & \blacksquare \\ & \blacksquare & \blacksquare \end{pmatrix} + xR\begin{pmatrix} \blacksquare & \blacksquare \\ & \blacksquare \end{pmatrix} \\
&= R\begin{pmatrix} \blacksquare & & \\ & & \blacksquare \\ & \blacksquare & \blacksquare \end{pmatrix} + xR\begin{pmatrix} \blacksquare & \blacksquare \end{pmatrix} + xR\begin{pmatrix} \blacksquare & \blacksquare \\ & \blacksquare \end{pmatrix} \\
&= R(\blacksquare)R\begin{pmatrix} \blacksquare & \blacksquare \\ & \blacksquare \end{pmatrix} + x(1+2x) + xR\begin{pmatrix} \blacksquare & \blacksquare \\ & \blacksquare \end{pmatrix} \\
&= (1+x)(1+3x+x^2) + x(1+2x) + x(1+3x+x^2) \\
&= (1+2x)(1+4x+x^2) \\
&= 1 + 6x + 9x^2 + 2x^3.
\end{aligned}$$

(iv) The forbidden positions board is $B = \begin{array}{cccccc} \blacksquare & \blacksquare & & & & \\ \blacksquare & & & & \blacksquare & \\ & & & \blacksquare & & \blacksquare \\ & \blacksquare & & & \blacksquare & \\ & & \blacksquare & & & \blacksquare \\ & & \blacksquare & \blacksquare & & \end{array}$. The boards B_1, formed from rows 1,2 and 4, and B_2, formed from rows 3,5 and 6, have no entries in the same column, so $R(B) = R(B_1)R(B_2)$. Now

$$B_1 = \begin{array}{ccc} \blacksquare & \blacksquare & \\ \blacksquare & & \blacksquare \\ & \blacksquare & \blacksquare \end{array} \quad \text{and} \quad B_2 = \begin{array}{ccc} & \blacksquare & \blacksquare \\ \blacksquare & & \blacksquare \\ \blacksquare & \blacksquare & \end{array}.$$

From the calculation in (ii) we have $R(B_1) = R(B_2) = 1 + 6x + 9x^2 + 2x^3$. Thus

$$\begin{aligned} R(B) &= (1 + 6x + 9x^2 + 2x^3)^2 \\ &= 1 + 12x + (9 + 36 + 9)x^2 + (2 + 54 + 54 + 2)x^3 + (12 + 81 + 12)x^4 + 36x^5 + 4x^6 \\ &= 1 + 12x + 54x^2 + 112x^3 + 105x^4 + 36x^5 + 4x^6. \end{aligned}$$

The forbidden positions rule gives

$$\begin{aligned} & 6! - 12 \times 5! + 54 \times 4! - 112 \times 3! + 105 \times 2! - 36 + 4 \\ = {} & 720 - 12 \times 120 + 54 \times 24 - 672 + 210 - 32 \\ = {} & \mathbf{82} \end{aligned}$$

possible choices for the third row.

$\boxed{5}$ (i) Write $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $C(x) = 1 - 3x + 2x^2$. Then $C(x)A(x) = (1 - 3x + 2x^2)A(x) = b_0 + b_1 x$. So $b_0 + b_1 x = (1 - 3x + \cdots)(1 + 4x + \cdots) = 1 + x$.

Hence

$$A(x) = \frac{1 + x}{1 - 3x + 2x^2} = \frac{1 + x}{(1 - x)(1 - 2x)} = \frac{A}{1 - x} + \frac{B}{1 - 2x}.$$

Partial fractions gives $A = -2, B = 3$, and $A(x) = -2\sum x^n + 3\sum 2^n x^n$, so $a_n = \mathbf{3 \times 2^n - 2}$.

(ii)

$$(1-4x+4x^2)A(x) = b_0 + b_1 x + \sum_{n=0}^{\infty}(a_{n+2} - 4a_{n+1} + 4a_n)x^{n+2}.$$

LHS $= (1-4x+\cdots)(1+x+\cdots) = 1-3x+\cdots$, so

$$(1-4x+4x^2)A(x) = 1-3x+\sum_{n=0}^{\infty} x^{n+2} = 1-3x+\frac{x^2}{1-x} = \frac{1-4x+4x^2}{1-x}.$$

Then $A(x) = \dfrac{1}{1-x}$ and $a_n = \mathbf{1}$.

(iii) Take $C(x) = 1-2x$. Then

$$(1-2x)A(x) = b_0 + \sum_{n=0}^{\infty}(a_{n+1} - 2a_n)x^{n+1} = 0 + \sum_{n=0}^{\infty} 3^{n+1}x^{n+1} = \frac{3x}{1-3x}.$$

Then

$$A(x) = \frac{3x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}.$$

Gives $A = -3, B = 3$ and so $a_n = -3 \times 2^n + 3 \times 3^n = \mathbf{3(3^n - 2^n)}$.

(iv) Again take $C(x) = 1-2x$.

$$(1-2x)A(x) = b_0 + \sum_{n=0}^{\infty}(a_{n+1} - 2a_n)x^{n+1} = 1 + \sum_{n=0}^{\infty} 2^{n+1}x^{n+1} = \frac{1}{1-2x}.$$

Then $A(x) = \dfrac{1}{(1-2x)^2}$, giving $a_n = \mathbf{(n+1)2^n}$.

(v) Here $C(x) = 1-3x$, giving

$$(1-3x)A(x) = b_0 + \sum_{n=0}^{\infty}(a_{n+1} - 3a_n)x^{n+1} = b_0 + 2\sum_{n=0}^{\infty}(n+1)x^{n+1}.$$

Now $b_0 = 0$, so RHS $= 2(x+2x^2+3x^3+\cdots) = \dfrac{2x}{(1-x)^2}$.

Then

$$A(x) = \frac{2x}{(1-x)^2(1-3x)}.$$

Write this as

$$\frac{2x}{(1-x)^2(1-3x)} = \frac{A}{1-3x} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

to get $2x = A(1-x)^2 + (1-3x)(B(1-x)+C)$ Put $x=1$ to get $C=-1$. Constant term gives $A+B+C=0$ while term in x^2 gives $A+3B=0$, so $C=2B$, and $B=-\frac{1}{2}$, $A=\frac{3}{2}$, $C=-1$.

Then $a_n = \frac{3}{2}3^n - \frac{1}{2} - (n+1) = \mathbf{\frac{1}{2}(3^{n+1}-1) - (n+1)}$.

$\boxed{\mathbf{6}}$ (i) $c_0 = 1,\ c_1 = 1,\ c_2 = c_0c_1 + c_1c_0 = 2,\ c_3 = c_0c_2 + c_1c_1 + c_2c_0 = 5,\ c_4 = 14,\ c_5 = 42$.

(ii) The constant term is 1 on both sides. For $n \geq 0$ the coefficient of x^{n+1} in RHS is the coefficient of x^n in $C(x)^2$ which is $\sum_{r=0}^{n} c_r c_{n-r} = c_{n+1}$. This is the same as the coefficient on LHS.

(iii) Number the points $1, \ldots, 2n$ around the circle. The line from point number 1 must have its other end at an even number, for otherwise there would be an odd number of points on each side of the line, which cannot be arranged in pairs without the line being crossed. If point 1 is joined to point $2r$, then there are $2(r-1)$ points on one side and $2(n-r)$ on the other. In each case, the problem of pairing up the points is the same as the original problem, but with a different value of n, so the numbers of solutions are a_{r-1} and a_{n-r}. Thus the total number of solutions with 1 joined to $2r$ is $a_{r-1}a_{n-r}$ for $1 < r < n$, and is a_{n-1} when $r=1$ or $r=n$. Then $a_n = \sum_{r=1}^{n} a_{r-1}a_{n-r}$, provided $n \geq 1$ taking $a_0 = 1$. This gives the same initial condition and recursive formula as the Catalan numbers, so $a_n = c_n$ for all n.

(iv) Write a_n for the number of ways of dissecting an $(n+2)$-gon into n triangles, setting $a_0 = 1$.

Label the vertices of $(n+2)$-gon $A_0, A_1, \ldots, A_{n+1}$ in order. Count the dissections in which the triangle with edge A_0A_{n+1} has third vertex at A_r.

There is an $(r+1)$-gon to the other side of A_0A_r which can be dissected in a_{r-1} ways, and an $(n-r+2)$-gon to the other side of A_rA_{n+1} which can be dissected in a_{n-r} ways, making a total of $a_{r-1}a_{n-r}$ ways, with the convention $a_0 = 1$ dealing with the cases $r = 1, n$. The total is then $a_n = \sum_{r=1}^{n} a_{r-1}a_{n-r}$. Then $a_{n+1} = \sum_{r=0}^{n} a_r a_{n-r}$ giving the same recursion and initial conditions as the Catalan numbers, so $a_n = c_n$.

(v) The formula for the quadratic $xC(x)^2 - C(x) + 1 = 0$ gives $C(x) = (1 \pm \sqrt{1-4x})/2x$ and so $xC(x) = \frac{1}{2}(1 \pm \sqrt{1-4x})$. RHS must give 0 when $x=0$, so $xC(x) = \frac{1}{2}(1-(1-4x)^{1/2})$.

Expanding $(1-4x)^{1/2}$ by the binomial theorem gives a sum where the coefficient of x^{n+1} is $\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\ldots\frac{1-2n}{2}(-4)^{n+1}/(n+1)!$. Cancelling the factors -2 leads to $-1.3.5.\ldots.(2n-1).2^{n+1}/(n+1)! = -2\frac{(2n)!}{n!(n+1)!}$. Divide by $-2x$ to obtain $C(x)$, and the formula for c_n follows. [20 marks]

$\boxed{7}$ Let s_n be the number of solutions of $n = a + 3b + 4c$ in non-negative integers. Then

$$\begin{aligned} S(t) = \sum_{n=1}^{\infty} s_n t^n &= (1 + t + t^2 + \ldots)(1 + t^3 + t^6 + \ldots)(1 + t^4 + t^8 + \ldots) \\ &= 1/(1-t)(1-t^3)(1-t^4). \end{aligned}$$

Then $S(t) = 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 5t^7 + 6t^8 + 7t^9 + 8t^{10} + \cdots$

The product $\prod_{i=1}^{\infty}(1-t^i)^{-1}$ is a sum of terms $t^{a_1}t^{2a_2}t^{3a_3}\cdots$; by associating this term to the partition with a_i parts equal to i for each i, we see that the terms which multiply out to t^n correspond exactly to the partitions of n, so this product is equal to $P(t)$.

We may argue similarly for partitions into parts of length at most m we restrict the product to $i \leq m$, so that $P_m(t) = \prod_{i=1}^{m}(1 - t^i)^{-1}$.

The Ferrers graph of a partition λ is obtained by arranging the parts of λ in descending order as $\lambda_1, \ldots, \lambda_k$ and then taking a row of λ_1 dots; below the first λ_2 of these a second row of dots, and so on.

The graph of a partition with all parts of length at most m has at most m columns. Interchanging rows and columns carries one Ferrers graph to another. It determines a bijection, which carries a graph with at most m columns to one with at most m rows, representing a partition with at most m parts. Hence the number of partitions of n with at most m parts is equal to the number of partitions of m with all parts at most m.

The generating function $R(t)$ for partitions with at most 4 parts is then the same as $P_4(t)$, namely $R(t) = \prod_{i=1}^{4}(1 - t^i)^{-1} = (1 - t^2)^{-1}S(t)$. So $R(t) = (1 + t^2 + t^4 + t^6 + t^8 + t^{10})S(t)$, up to terms in t^{10}. Thus $R(t) = 1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 + 18t^9 + 23t^{10} + \ldots$.

The corresponding generating function for partitions with at most 3 parts is $P_3(t)$. Now $P_3(t)(1 - t^4)^{-1} = P_4(t) = R(t)$ so $P_3(t) = (1 - t^4)R(t) = 1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + 8t^7 + 10t^8 + 12t^9 + 14t^{10} + \ldots$.

Partitions with exactly 4 parts will be counted by $P_4(t) - P_3(t)$ and hence by $t^4 + t^5 + 2t^6 + 3t^7 + 5t^8 + 6t^9 + 9t^{10}$ up to $n = 10$.

There are thus **9** partitions of 10 with exactly 4 parts. These are (7,1,1,1), (6,2,1,1), (5,3,1,1), (4,4,1,1), (5,2,2,1), (4,3,2,1), (4,2,2,2), (3,3,2,2), (3,3,3,1), with Ferrers graphs

$$\begin{array}{lllllll} \times & \times & \times & \times & \times & \times & \times \\ \times &&&&&& \\ \times &&&&&& \\ \times &&&&&& \end{array}, \quad \begin{array}{llllll} \times & \times & \times & \times & \times & \times \\ \times & \times &&&& \\ \times &&&&& \\ \times &&&&& \end{array}, \quad \begin{array}{lllll} \times & \times & \times & \times & \times \\ \times & \times & \times && \\ \times &&&& \\ \times &&&& \end{array}, \quad \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times &&& \\ \times &&& \end{array},$$

$$\begin{array}{lllll}\times\times\times\times\times\\ \times\times\\ \times\times\\ \times\end{array},\quad \begin{array}{llll}\times\times\times\times\\ \times\times\times\\ \times\times\\ \times\end{array},\quad \begin{array}{llll}\times\times\times\times\\ \times\times\\ \times\times\\ \times\times\end{array},\quad \begin{array}{lll}\times\times\times\\ \times\times\times\\ \times\times\\ \times\times\end{array},\quad \begin{array}{lll}\times\times\times\\ \times\times\times\\ \times\times\times\\ \times\end{array}$$

[20 marks]

8 (a) A function of N variables $\{x_1, \ldots, x_N\}$ is called *symmetric* if it is unchanged by any permutation of the x_i. The *elementary symmetric function* σ_n is the sum of all products $x_{i_1} \ldots x_{i_n}$ with $1 \le i_1 < \ldots < i_n \le N$. The *power sum symmetric function* π_n is the sum $\sum_{i=1}^N x_i^n$. Let us also write $\sigma_0 = 1$, $\pi_0 = N$.

The Newton Identities state that $r\sigma_r = \sum_{i=0}^r (-1)^{i-1}\sigma_{r-i}\pi_i$ for all r, taking $\sigma_r = 0$ when $r > N$.

To prove them, write $E(t) = \sum_{r=0}^N \sigma_r t^r = \prod_{i=1}^N (1 + x_i t)$.

Then $\ln(E(t)) = \sum_{i=1}^N \ln(1 + x_i t)$.

Differentiate with respect to t to get

$$\begin{aligned}\frac{E'(t)}{E(t)} = \sum_{i=1}^N \frac{x_i}{1 + x_i t} &= \sum_{i=1}^N \left(x_i \sum_{m=0}^\infty (-x_i t)^m \right)\\ &= \sum_{m=0}^\infty \left(\sum_{i=1}^N x_i(-x_i)^m \right) t^m = \sum_{m=0}^\infty (-1)^m \left(\sum_{i=1}^N x_i^{m+1} \right) t^m\\ &= \sum_{m=0}^\infty (-1)^m \pi_{m+1} t^m.\end{aligned}$$

Multiply up by $E(t)$ to get $E'(t) = \sigma_1 + 2\sigma_2 t + \cdots + r\sigma_r t^{r-1} + \cdots = (\pi_1 - \pi_2 t + \pi_3 t^2 - \cdots)(1 + \sigma_1 t + \sigma_2 t^2 + \cdots)$.

Equate the coefficients of t^{r-1} for each r to get Newton's identities (Newton's relations).

(b) (i) The polynomial $(x - \alpha)(x - \beta)(x - \gamma)$ has the form $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$ where σ_i are the elementary symmetric functions of α, β, γ.

So here $\alpha + \beta + \gamma = 2$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = -5$.

From the first three Newton identities $\pi_1 = \sigma_1$, $\pi_2 - \pi_1\sigma_1 = -2\sigma_2$, $\pi_3 - \pi_2\sigma_1 + \pi_1\sigma_2 = 3\sigma_3$, we deduce in turn $\pi_2 = \sigma_1^2 - 2\sigma_2 = 4 + 2 = 6$, $\pi_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 = 8 + 6 - 15 = -1$.

(ii) Put $x_1 = \alpha^2, x_2 = \beta^2, x_3 = \gamma^2$. The required polynomial is then $x^3 - s_1 x^2 + s_2 x - s_3 = 0$, where $s_1 = x_1 + x_2 + x_3, s_2 = x_1x_2 + x_2x_3 + x_3x_1$ and $s_3 = x_1x_2x_3$.

Now

$s_1 = 6$
$s_2 = (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha + \beta + \gamma)(\alpha\beta\gamma) = 1 + 20$
$s_3 = (\alpha^2\beta^2\gamma^2) = (\alpha\beta\gamma)^2 = 25.$

The polynomial is then $x^3 - 6x^2 + 21x - 25$.