Solutions for 344 exam

1ai. Algebraic manipulation:

$$\begin{aligned}\binom{a}{b}\binom{b}{c} &= \frac{a!b!}{b!(a-b)!c!(b-c)!} \\ &= \frac{a!(a-c)!}{c!(a-c)!(a-b)!(b-c)!} \\ &= \binom{a}{c}\binom{a-c}{b-c}.\end{aligned}$$

Combinatorial argument: suppose given a people. We show that the two sides of the given equality both count the number of ways to form a committee of b of them and a subcommittee of c of those. On the one hand, there are $\binom{a}{b}$ ways to make the committee, and $\binom{b}{c}$ ways to choose c of its members as the subcommittee. On the other hand, we could choose the subcommittee first, in $\binom{a}{c}$ ways, and the $b-c$ members of the committee who are not on the subcommittee from the remaining $a-c$ people: that is, in $\binom{a-c}{b-c}$ ways. [lecture] [4 marks]

1aii. Algebraic manipulation: consider the equality $(1+x)^{2n} = (1+x)^n(1+x)^n$. Comparing coefficients of x^n on both sides, we get $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$. But $\binom{n}{i} = n!/(i!(n-i)!) = \binom{n}{n-i}$, so this is the desired equality. Combinatorial argument: suppose given n men and n women. We show that the two sides both count the number of ways to choose n of them. On the one hand, we could ignore the distinction and simply choose n of the people. This can be done in $\binom{2n}{n}$ ways. On the other hand, we could maintain the distinction: we choose n women and 0 men, or $n-1$ women and 1 man, etc., or 0 women and n men. Thus the total is $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$, which is equal to $\sum_{i=0}^{n}(\binom{n}{i})^2$ as in the previous solution. [homework] [4 marks]

1b. Let S_1 be the set of ways to choose 13 cards including 3 diamonds, S_2 the set of ways that include 4 hearts, and S_3 the set of ways including 5 spades. Then, by the principle of inclusion-exclusion, the desired number is

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_3 \cap S_1| + |S_1 \cap S_2 \cap S_3|.$$

Now, $|S_1|$ is the number of ways to choose 3 diamonds and 10 non-diamonds, so it is $\binom{13}{3}\binom{39}{10}$. Similarly $|S_2| = \binom{13}{4}\binom{39}{9}$ and $|S_3| = \binom{13}{5}\binom{39}{8}$. On the other hand, $|S_1 \cap S_2|$ is the number of ways to choose 3 diamonds, 4 hearts, and 6 black cards, so it is $\binom{13}{3}\binom{13}{4}\binom{26}{6}$, etc., and likewise $|S_1 \cap S_2 \cap S_3| = \binom{13}{3}\binom{13}{4}\binom{13}{5}\binom{13}{1}$. The answer is therefore

$$\binom{13}{3}\binom{39}{10} + \binom{13}{4}\binom{39}{9} + \binom{13}{5}\binom{39}{8} - \binom{13}{3}\binom{13}{4}\binom{26}{6} \\ - \binom{13}{3}\binom{13}{5}\binom{26}{5} - \binom{13}{4}\binom{13}{5}\binom{26}{4} + \binom{13}{3}\binom{13}{4}\binom{13}{5}\binom{13}{1}.$$

(It is not necessary to simplify this in any way.) [lecture and homework] [6 marks]

1c. This word has 2 N's, 3 D's, 3 E's, and two unique letters, so the number of ways is $\binom{10}{3,3,2,1,1} = 10!/(3!\,3!\,2!\,1!\,1!)$. First we find the number of rearrangements not including NN: we may think of NN as a single letter, so the number of ways including NN is $9!/(3!\,3!)$ (ignoring $1! = 1$ in the denominator), and the number of rearrangements not including it is $10!/(3!\,3!\,2!) - 9!/(3!\,3!)$. Now, for EE, the simplest way is the following. The other 7 letters can be arranged in $7!/(3!\,2!\,1!\,1!)$ ways. Then there are 8 positions between and around the other 7 letters, and we get arrangements with nonadjacent E's by putting them in distinct positions, which can be done in $\binom{8}{3}$ ways. So the answer is $\binom{8}{3}\binom{7}{3,2,1,1}$. [similar to lecture and homework] [6 marks, 2 each]

2a. For $1 \le i \le b$, let $n_i = \sum_{j=1}^{i} m_b$, and let $p_i = n_i + c$. Then all the n_i and p_i are positive integers less than or equal to $a+c$, and there are $2b$ of them, so by the pigeonhole principle two of them must be equal. Because all of the m_i are positive, it is not possible for two of the n_i or two of the p_j to be equal, so we must have $n_i = p_j$ for some i, j. But then, by definition, we have $c = n_i - n_j = \sum_{k=j+1}^{i} m_k$. [homework] [5 marks]

2b. Dissect the square into four squares of side length $1/2$ by drawing the lines that join the midpoints of opposite sides. By the pigeonhole principle, two of the points must lie in or on the boundary of the same square. It is clear that the distance between two points in or on a square is no greater than the length of the diagonal, so the distance between those two points is at most $1/\sqrt{2}$ (Pythagorean theorem). On the other hand, if we choose the five points to be the four corners and the centre, the minimum distance is exactly $1/\sqrt{2}$. [lecture] [6 marks]

2c. T has 128 subsets, and the possible sums range from 0 to $19 \cdot 7 = 133$. If there is a subset with sum 127 or greater, then the smallest element of T must be at least 10, so the 127 sums of nonempty subsets are in the interval $[10, 133]$, which contains only 124 integers, so two of them must be equal. If not, then the 128 sums are all in the range $[0, 126]$, and again two of them are equal. (22 is nowhere near best possible, of course, but it seems difficult to determine what is.) [similar to lecture] [9 marks]

3a. Hall's theorem says that if S and T are finite sets of the same size n and each element of S can be matched to some elements of T, then it is possible to match every element of S to an element of T if and only if, for every subset R of size k of S, the total number of elements of T that can be matched to at least one element of R is at least k. More generally, if elements of S may be paired with multiple elements of T, the theorem states that a matching exists if and only if, for every subset R of S, the total number of elements of T that can be matched to at least one element of R is at least the total number of matches required. [lecture] [4 marks]

3bi. If we apply the standard algorithm, we might assign successively 1 to A, 4 to B, 2 to C, 5 to D, 6 to E, 3 to F, and then notice that G and H cannot be assigned any task. But G or H could do $1, 5, 6$, which are being done by A, D, E. These people can do $1, 3, 5, 6$, some of which could be done by B or F, who can do 7 or 8. So we could take 6 from E and give it to G, then take 3 from F and give it to E, and then assign task 8 to F. Thus seven tasks can be assigned. But if we attempt to give H a task, we notice that H can do 1 and 5, which are being done by A and D, who can do $1, 3, 5, 6$, which are done by A, D, E, G, and there the process stops. So A, D, E, G, H can do only four tasks between them, and the assignment is not possible. (Other correct answers are possible.) [similar to lecture and homework] [8 marks]

3bii. We create new people B_1, B_2 replacing B, both of whom can do tasks $1, 4, 7, 8$. Starting with the assignment $A1, B_14, C2, D5, E3, F8, G6$ from the previous paragraph, we see that B_2 can immediately be assigned task 7, solving the problem. [similar to lecture and homework] [3 marks]

3biii. Again, let us start with the assignments of the previous part: $A1, B_14, B_27, C2, D5, E3, F8$ (having deleted G). Now H can do $1, 5$, which are done by A, D, who can do $1, 3, 5, 6$. So we want to give task 6 to D, which frees task 5 to be assigned to H. [similar to lecture and homework] [5 marks]

4i. A stable matching is a permutation π of $\{1, 2, 3, 4\}$ such that there do not exist $i, j \in \{1, 2, 3, 4\}$ such that j is higher on x_i's preference list than $\pi(i)$ and i is higher on y_j's preference list than $\pi^{-1}(j)$. (It is acceptable to use more words and fewer symbols in the explanation.) [lecture] [2 marks]

4ia. Using the standard algorithm, we have the following steps, with tentative assignments as shown:

Proposal	Status
(x_1, y_3)	3...
(x_2, y_2)	32..
(x_3, y_2)	3.2.
(x_2, y_1)	312.
(x_4, y_3)	.123
(x_1, y_2)	21.3
(x_3, y_4)	2143

and a stable matching is achieved: $(x_1, y_2), (x_2, y_1), (x_3, y_4), (x_4, y_3)$. [lecture and homework] [4 marks]

4ib. Similarly, we have the following steps:

Proposal	Status
(y_1, x_1)	1...
(y_2, x_4)	14..
(y_3, x_2)	142.
(y_4, x_1)	.421
(y_1, x_3)	3421

which produces the stable matching $(x_1, y_4), (x_2, y_3), (x_3, y_1), (x_4, y_2)$. [lecture and homework] [4 marks]

4iia. To set up this problem, we first remove x_4 and y_1. We then remove y_j from the preference list of x_i if x_i prefers y_1 to y_j and y_1 prefers x_i to x_4; then we remove x_k from the preference list of y_l if y_l prefers x_4 to x_k and x_4 prefers y_l to y_1. In particular, this means we remove x_1 and x_3 from y_3's list. Then we introduce an imaginary student x_0 and an imaginary residency y_0, such that x_0 is at the bottom of the list for all of the y_i and y_0 is at the bottom for all the x_i, and then we add the deleted x_1 and x_3 to the end of y_3's list. A solution of the original problem exists if and only if all solutions of the new problem have x_0 matched with y_0, if and only if one solution has this property. The preference lists might now read:

0	2	3	4	0
1	3	2	4	0
2	2	3	4	0
3	2	4	3	0

0	1	2	3	0
2	1	3	2	0
3	2	0	3	1
4	1	3	2	0

Again, applying our standard solution method, we would have successively:

Proposal	Status
(x_0, y_2)	2...
(x_1, y_3)	23..
(x_2, y_2)	.32.
(x_0, y_3)	3.2.
(x_1, y_2)	32..
(x_2, y_3)	.23.
(x_0, y_4)	423.
(x_3, y_2)	423.
(x_3, y_4)	.234
(x_0, y_0)	0234

Thus there is such a stable matching, namely $(x_1, y_2), (x_2, y_3), (x_3, y_4), (x_4, y_1)$. [similar to lecture and homework] [6 marks]

4iib. There is no such matching. We have already found that in the student-pessimal solution x_4 is paired with y_2. But x_4 prefers y_2 to y_4, so it is not possible to find a stable matching where x_4 is paired with y_4. (Alternatively, the method of 4(ii)(a) could be applied, or one could use the theorem that if there is a stable matching in which x_i and y_j are matched despite each being the other's last choice, then this is true of all stable matchings.) [similar to lecture and homework] [4 marks]

5i. Given a board consisting of some of the squares of an $m \times n$ rectangle, the rook polynomial is the polynomial $\sum r_k t^k$, where r_k is the number of ways to place k nonattacking rooks on the board. It is related to the rook polynomial of the complementary board by the formula

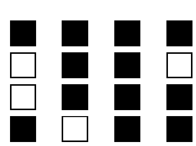
$$\overline{r}_k = \sum_{j=0}^{k} (-1)^j c_{m-j,n-j,k-j} r_j,$$

where $\overline{r}_k$ is the kth coefficient of the complementary board and $c_{r,s,t}$ is the number of ways to place t nonattacking rooks on an entire $r \times s$ board, which is

$$\frac{r!\,s!}{(r-t)!\,(s-t)!\,t!}.$$

[lecture] [6 marks]

5ii. We use the formula given above. The complementary board is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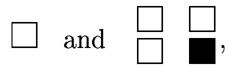
its rook polynomial is the product of those of

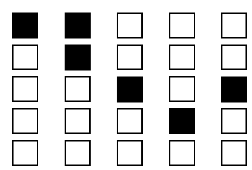


so $(1+t)(1+3t+t^2) = 1+4t+4t^2+t^3$. Using the formula above to determine the rook polynomial of the complementary board, we find

$$\begin{aligned}
\overline{r}_0 &= c_{4,4,0} = 1 \\
\overline{r}_1 &= c_{4,4,1} - 4c_{3,3,0} = 12 \\
\overline{r}_2 &= c_{4,4,2} - 4c_{3,3,1} + 4c_{2,2,0} = 72 - 36 + 4 = 40 \\
\overline{r}_3 &= c_{4,4,3} - 4c_{3,3,2} + 4c_{2,2,1} - c_{1,1,0} = 96 - 72 + 16 - 1 = 39 \\
\overline{r}_4 &= c_{4,4,4} - 4c_{3,3,3} + 4c_{2,2,2} - c_{1,1,1} = 24 - 24 + 8 - 1 = 7.
\end{aligned}$$

Therefore the answer is $1 + 12t + 40t^2 + 39t^3 + 7t^4$. [similar to homework] [8 marks]

5iii. This question amounts to determining the coefficient of t^5 in the rook polynomial of the board



The complementary board has rook polynomial $(1+3t+t^2)(1+2t)(1+t) = 1+6t+12t^2+9t^3+2t^4$. So by the formula given before, the desired number is

$$c_{5,5,5} - 6c_{4,4,4} + 12c_{3,3,3} - 9c_{2,2,2} + 2c_{1,1,1} = 120 - 144 + 72 - 18 + 2 = 32.$$

[similar to lecture and homework] [6 marks]

6ai. Let $f(t) = \sum a_i t^i$. Then $f(t)(4t^2 - 4t + 1) = a_0 + (a_1 - 4a_0)t = 2 - 6t$. It follows that $f(t) = (2-2t)/(4t^2-4t+1)$. Now, $4t^2 - 4t + 1 = (1-2t)^2$, so we write $f(t) = -1/(1-2t)^2 + 3/(1-2t)$. Each of these has a known expansion: $1/(1-2t) = \sum 2^i t^i$ and $1/(1-2t)^2 = \sum 2^i(i+1)t^i$. So $f(t) = \sum 2^i(2-i)t^i$, and the explicit formula is $a_i = 2^i(2-i)$. [similar to lecture] [4 marks]

6aii. From the first equation we have $b_i = a_{i+1} - 2a_i$ and $b_{i-1} = a_i - 2a_{i-1}$. Substituting these into the second equation, we get

$$a_{i+1} - 2a_i = a_{i-1} + 2a_i - 4a_{i-1},$$

which rearranges to $a_{i+1} = 4a_i - 3a_{i-1}$. Noting that $a_1 = 2\cdot 1 - 1 = 1$ we see, as before, that $f(t) = \sum a_i t^i$ has the property that $f(t)(3t^2 - 4t + 1) = 1 - 3t$, or in other words $f(t) = 1/(1-t)$. Thus in fact $a_i = 1$ for all i. (It is acceptable to notice this and prove it directly by induction.) [similar to lecture and homework] [4 marks]

6aiii. Again letting $f(t) = \sum a_i t^i$, we have $f(t)(1-3t) = 1 + \sum_1^\infty 2^i t^i = 1/(1-2t)$. Thus $f(t) = 1/(1-2t)(1-3t) = 3/(1-3t) - 2/(1-2t)$, so that $a_i = 3^{i+1} - 2^{i+1}$ for all i. [similar to lecture and homework] [5 marks]

6b. We derive a recurrence relation for a_i by expanding the determinant along the top row. The determinant of the $1,1$ minor is a_{i-1}; that of the $1,2$ minor is $-a_{i-2}$, as is seen by expanding that minor along the first column. By basic properties of determinants it follows that $a_i = 2a_{i-1} + a_{i-2}$. One checks directly that $a_1 = 2$, so as before we have $f(t)(t^2 + 2t - 1) = -1$ and $f(t) = -1/(t^2 + 2t - 1)$. For the explicit formula, let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of $t^2 - 2t - 1$. Then the explicit formula will be $a_i = c\alpha^i + d\beta^i$ for some constants c, d. We have $c + d = 1$ and $c\alpha + d\beta = 2$, so that $c\alpha + (1-c)\beta = 2$, or $c = (2-\beta)/(\alpha - \beta) = (2+\sqrt{2})/4$. Thus $d = (2-\sqrt{2})/4$. [similar to lecture] [7 marks]

7. The generating functions are respectively:
   (a) $\prod_{2\nmid i} \frac{1}{1-t^i} \prod_{2|j}(1+t^j)$,
   (b) $\prod_{4\nmid i} \frac{1}{1-t^i}$,

(c) $\prod_i (1 + t^i + t^{2i} + t^{3i})$.

(No work need be shown for any of these.) [lecture] [3 marks each]

To see that the function in (b) is equal to that in (a), start by cancelling $\prod_{2\nmid i} \frac{1}{1-t^i}$ from both sides. We must then prove that $\frac{1}{(1-t^2)(1-t^6)(1-t^{10})\ldots} = (1+t^2)(1+t^4)\ldots$. Substituting $u = t^2$, we must prove that $\prod_{2\nmid i} \frac{1}{1-u^i} = \prod 1 + u^i$. But $1 + u^i = (1-u^{2i})/(1-u^i)$, and in $\prod \frac{1-u^{2i}}{1-u^i}$ we may cancel all terms with even exponent from the denominator to get $\prod_{2\nmid i} \frac{1}{1-u^i}$ as desired. Next, to see that the function in (b) is equal to that in (c), simply multiply numerator and denominator of (b) by $\prod_{4|i}(1-t^i)$. This gives $\prod_i \frac{1-t^{4i}}{1-t^i}$, and that is certainly equal to the function in (c). [lecture or homework] [7 marks for showing that any two of them are equal; 11 for showing that all three are]

8i. Newton's identities say the following: let $x_1, \ldots, x_n$ be any numbers. Let r_k be the sum of all products of k distinct x_i, where $r_0 = 1$ and $r_k = 0$ if $k > n$; let $s_k = \sum_1^n x_i^k$. Then, for all nonnegative integers m, we have

$$mr_m = \sum_1^m (-1)^{j+1} s_j r_{m-j}.$$

[4 marks]

For these numbers, we have $s_1 = 3, s_2 = 5, s_3 = 7$, and we must determine r_1, r_2, r_3, s_4, s_5. Indeed, $r_1 = s_1 = 3$. By Newton's identities above, we have $2r_2 = s_1r_1 - s_2r_0$, which $r_2 = (9-5)/2 = 2$. For r_3, we get $3r_3 = s_1r_2 - s_2r_1 + s_3r_0$, so $r_3 = (6 - 15 + 7)/3 = -2/3$. Therefore the polynomial is $x^3 - 3x^2 + 2x - 2/3$. Then, clearly $r_4 = r_5 = 0$, so again applying Newton's identities we get $0 = s_1r_3 - s_2r_2 + s_3r_1 - s_4r_0$, so $s_4 = -2 - 10 + 21 = 9$, and $0 = s_1r_4 - s_2r_3 + s_3r_2 - s_4r_1 + s_5r_0$, so $s_5 = -0 - 10/3 - 14 + 27 = 29/3$. [lecture and homework] [10 marks]

Solutions that avoid Newton's identities, instead solving directly for the elementary symmetric functions in terms of the power sums, will be given 1 mark for r_1, 2 for r_2, 3 for r_3, and 4 each for s_4 and s_5.

8ii. The sum of the roots is $2(a+b+c) = 2r_1 = 6$; the sum of products of pairs is $3(ab+bc+ca)+a^2+b^2+c^2 = (a+b+c)^2 + (ab+bc+ca) = r_1^2 + r_2 = 11$; and the product of the roots is

$$\begin{aligned}
&2abc + a^2(b+c) + b^2(a+c) + c^2(a+b) \\
&\qquad = 2r_3 + (a+b+c)(a^2+b^2+c^2) - (a^3+b^3+c^3) \\
&\qquad = 2r_3 + r_1s_2 - s_3 \\
&\qquad = -4/3 + 15 - 7 = 20/3.
\end{aligned}$$

Thus the desired polynomial is $x^3 - 6x^2 + 11x - 20/3$. [lecture and homework] [6 marks]