1. (a) A party of 24 people regularly travel together to the theatre.

(i) Six people in the party drive their own cars and each takes three further passengers. Having regard simply to which people travel together, find how many different travel arrangements there are for the party.

(ii) Occasionally one of the drivers is unable to go, and everybody travels in five cars, with 4 passengers in three of them and 3 in the other two. Assuming that all the cars are able to carry 4 passengers (besides the driver) find how many travel arrangements are possible in these conditions.

(iii) For a gala evening they decide to leave their cars at home and book a fleet of six taxis, each carrying 4 passengers. How many travel arrangements are possible, still having regard only to which people in the party travel together,

(1) assuming that all 24 in the party travel,

(2) assuming that one member of the party is unable to go?

[All answers should include a brief justification of the calculations].

(b) (i) Let A be any selection of k numbers from $\{1, 2, \ldots, 30\}$. Prove that there must be two different (possibly overlapping) subsets of 3 numbers in A with the same sum, when $k \geq 9$.

Prove also that there must be three different (possibly overlapping) subsets of 3 numbers in A with the same sum, when $k \geq 11$.

(ii) Prove that if $A = \{1, 2, 4, 8, 16\}$ then no two subsets of A, of any size, have the same sum.

[20 marks]

2. (i) Derive a formula for the number of ways of distributing r identical objects into n distinct containers.

Show why this formula also counts the number of solutions to the equation

$$k_1 + k_2 + \cdots + k_n = r$$

for non-negative integers $k_1, \ldots, k_n$.

(ii) Write down the coefficient of the monomial $x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ in the expansion of $(x_1 + \cdots + x_n)^r$.

Write down also the number of different monomials which appear in this expansion.

(iii) The number of anagrams of the word ABRACADABRA is the coefficient of some monomial in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{11}$. Identify one such monomial, and give its coefficient.

Find the number of anagrams in which AA does not occur.

(iv) State the inclusion-exclusion formula.

Find the number of integer solutions of $a + b + c + d = 24$, with

$$0 \leq a \leq 14,\ 0 \leq b \leq 11,\ 0 \leq c \leq 8,\ 0 \leq d.$$

[20 marks]

3. State Hall's Assignment Theorem.

(a) A rectangular array of squares is given, from which a subset B of the squares is excluded.

(i) Show how the problem of constructing a perfect cover of the remaining squares by 2×1 tiles can be reformulated in terms of selecting distinct representatives for a suitable collection of sets, describing carefully what sets are to be used.

(ii) For each array below, where excluded squares are shown as ■, list a suitable collection of sets for this reformulation.

(iii) Using this collection of sets decide, with reasons, whether or not there is a perfect cover of the squares shown as □ by 2×1 tiles, in each of the two cases shown.

```
□ □ ■ ■ □ ■        ■ □ □ □ □ □
□ □ □ ■ □ □        ■ ■ □ □ □ □
□ □ ■ ■ □ □        □ ■ ■ □ ■ ■
□ ■ □ □ □ □        □ □ □ □ ■ □
□ □ □ ■ □ □        □ □ □ □ □ □
```

(b) In its new premises a company has a suite of 8 adjacent offices $1, \ldots, 8$ available for its senior executives $A, B, \ldots, G$.

Taking into account the facilities in each office and the personal preferences of the executives, the only offices suitable for each executive are listed in the table below.

Executive	Office	Executive	Office
A	$1, 2, 6, 8$	E	$1, 5$
B	$2, 5, 8$	F	$6, 7$
C	$2, 3, 4$	G	$1, 2, 5$
D	$1, 4$		

Initially, executives $A, \ldots, F$ are allocated offices $1, \ldots, 6$ in order, but it is then realised that no appropriate office is left for G, the managing director.

Show how to reallocate the offices, without moving the company secretary, A, so that none of the executives share an office.

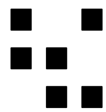
Suppose that A subsequently leaves the company, and his place is filled by H, who occupies office 1. Some time later A is headhunted to return to a new position which requires him to have an office next to G. Show that this cannot be done if G and H are to remain in their current offices, ignoring A's original preferences.

Show how to allocate offices, with H moving to office 2, so that A and G have adjacent offices, and all original preferences are respected.

[Justify your answers clearly.] [20 marks]

4. (i) Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula for an $m \times n$ board with $m \leq n$.

(ii) Calculate the rook polynomial of the 3×3 board shown.



(iii) A party of five friends have gone on an activity holiday which offers sessions of Archery, Bowls, Cycling, Diving, Equestrianism and Fencing.

(1) Find how many ways there are for the friends to arrange a session, each choosing a different activity, so as to satisfy their preferences as listed here: James and William will not try Bowls; Helen and Richard don't want to dive, nor do they want to fence; Sarah and James dislike horses, so Equestrianism is out for them, while William and Sarah don't fancy Archery.

(2) With the same restrictions, find how many ways a session can be arranged, on a day when cycling is not available.

[20 marks]

5. (a) Solve the following recurrence relations. In each case you should find an expression for a_n and also an expression for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 3a_{n+1} - 2a_n, n \geq 0,\ a_0 = 1, a_1 = 4.$

(ii) $a_{n+1} = 3a_n + 2^n, n \geq 0,\ a_0 = 1.$

(iii) $a_{n+2} = 2a_{n+1} - a_n, n \geq 0,\ a_0 = 1, a_1 = 2.$

(b) Let B_n be the $n \times n$ matrix with entries 5 on the diagonal, 2 immediately above the diagonal and 3 immediately below the diagonal, all other entries being 0. Write $b_0 = 1$ and $b_n = \det(B_n), n \geq 1$, so that, for example,

$$b_5 = \det \begin{pmatrix} 5 & 2 & 0 & 0 & 0 \\ 3 & 5 & 2 & 0 & 0 \\ 0 & 3 & 5 & 2 & 0 \\ 0 & 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 3 & 5 \end{pmatrix}.$$

Find a recurrence relation for b_{n+2} in terms of b_{n+1} and b_n. Hence find an expression for $B(x) = \sum_{n=0}^{\infty} b_n x^n$, and an expression for b_n. [20 marks]

6. The Catalan numbers $\{c_n\}$ are defined recursively by the relations

$$c_0 = 1, c_{n+1} = \sum_{r=0}^{n} c_r c_{n-r}, \;\; n \geq 0.$$

(i) Write down the sequence of these numbers up to c_5.

(ii) Prove, by comparing the terms in x^{n+1}, that the generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

for the Catalan numbers satisfies the quadratic equation

$$C(x) = 1 + xC(x)^2.$$

(iii) Write a_n for the number of ways to join $2n$ points on the circumference of a circle in pairs by n straight lines drawn across the circle which do not intersect each other. Find a recurrence relation expressing a_{n+1} in terms of $a_i, i \leq n$ and deduce that $a_n = c_n$ for all $n \geq 1$.

(iv) Write down the coefficient of y^n in the binomial series expansion of $(1+y)^\alpha$. Hence show that the coefficient of x^{n+1} in the expansion of $(1-4x)^{\frac{1}{2}}$ is $-2^{n+1}\dfrac{1 \times 3 \times \cdots \times (2n-1)}{(n+1)!}$.

(v) Using the equation $C(x) = 1 + xC(x)^2$ show that

$$xC(x) = \frac{1}{2}(1 - \sqrt{1-4x}),$$

and deduce that

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

[20 marks]

7. Obtain a formula for the generating function $S(t) = \sum_{n=1}^{\infty} s_n t^n$, where s_n is the number of solutions of $n = 2a + 3b + 4c$ in non-negative integers.

Calculate $S(t)$ up to the term in t^{10}.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer n. Find also the function $P_m(t)$ which enumerates the number of partitions of n into parts of length $\leq m$.

Define the term *Ferrers graph*. Use Ferrers graphs to establish a bijection between partitions of n with at most m parts and partitions of n into parts of length $\leq m$.

Hence write down the generating function $R(t)$ which enumerates partitions of n with at most 4 parts.

Show that $R(t) = (1-t)^{-1}S(t)$ and hence find the number of partitions of 10 with at most 4 parts.

Show that $(1-t^4)R(t)$ enumerates partitions of n with at most 3 parts.

Use your calculations to determine the number of partitions of n with exactly 4 parts for all $n \leq 10$. Exhibit the corresponding Ferrers graphs in the case $n = 9$.

[20 marks]

8. (a) Define the term *symmetric function*. For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

(b) (i) Obtain the equation with roots $\alpha\beta,\ \beta\gamma,\ \gamma\alpha$ and the equation with roots $\alpha^2,\ \beta^2,\ \gamma^2$ in terms of the elementary symmetric functions of α, β and γ.

(ii) Write d_r for the determinant

$$\det \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^r & \beta^r & \gamma^r \end{vmatrix}.$$

Show that, for each $r \geq 2$, $\phi_r = d_r/d_2$ is a symmetric function of α, β and γ.

Show that $d_2 = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$. Find d_3 and express ϕ_3 in terms of the elementary symmetric functions of α, β and γ.

[20 marks]