1. (i) In how many ways can you make an **unordered** selection of six integers from the list $\{-4, -3, -2, -1, 1, 2, 3, 4, 5\}$, assuming that repetitions are allowed? In how many ways if the integers are distinct?

(ii) For how many unordered selections (with repetitions) is the product of the six integers positive?

(iii) How many unordered selections of distinct numbers are there whose product is positive?

(iv) Show that any selection S of six distinct integers from the list above must contain at least three subsets of integers having the same sum.

[All answers should include a brief justification of the calculations.]

[20 marks]

2. (i) Derive a formula for the number of ways of distributing r identical objects into n distinct containers.

Show why this formula also counts the number of solutions to the equation

$$k_1 + k_2 + \cdots + k_n = r$$

for non-negative integers $k_1, \ldots, k_n$.

(ii) Write down the coefficient of the monomials x^2y^4t and $xy^2z^2t^2$ in the expansion of $(x + y + z + t)^7$.

Write down also the number of different monomials which appear in this expansion.

(iii) Find the number of solutions for the simultaneous equations $x_1 + x_2 + x_3 = 4$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 11$ in non-negative integers $x_1, \ldots, x_5$.

Find also the number of solutions when the equations are replaced by the inequalities $x_1 + x_2 + x_3 \leq 4$ and $x_1 + x_2 + x_3 + x_4 + x_5 \leq 11$.

(iv) Find the number of integer solutions of $a + b + c + d = 24$, with

$$0 \leq a \leq 13,\ 0 \leq b \leq 11,\ 0 \leq c \leq 10,\ 0 \leq d \leq 9.$$

[20 marks]

3. State Hall's Assignment Theorem.

(a) An amateur dramatic society has a wardrobe of 9 costumes for the characters in their next production. In deciding the cast the director has to choose actors who fit the costumes, as there are no funds available for alterations. At the initial fitting it was found that costumes for characters $1, \ldots, 9$ would fit actors $A, B, \ldots, I$, in that order. However a serious problem emerged in that Charles, who had been fitted up with costume 3, refused absolutely to play the corresponding part of the fairy godmother. Some other actors also declared that they would not play certain roles, even if they fitted the costumes for the part. The producer then drew up a table of actors who were suitable for each part and who would fit into its costume, as shown.

Character	*Actors*	*Character*	*Actors*
1	A, B, C, F	2	A, B, D, E, F
3	A, D, I	4	B, C, D, E, H
5	A, D, G	6	E, F, G, H
7	D, G	8	A, G, I
9	D, I		

Show that the producer can arrange to cast 8, but no more, of the characters under these restrictions.

Suppose that the producer now finds another member of the society who is able to play characters 1,3, or 6. Decide whether he can now find suitable actors for all 9 characters, and whether the gift of one further costume, which can be used by A, E or G for an extra role, will allow all 10 of the actors to take part.

(b) A cube shaped box has sides of length 3. You are given a supply of blocks of size $2 \times 1 \times 1$. Using an alternately black and white colouring of the 27 unit cubes into which the box can be divided construct a family of sets such that choices of distinct representatives correspond to ways of packing 13 blocks into the box, leaving a blank cube in the middle of the top face, and hence show that such a packing is possible.

Determine also whether or not it is possible to pack the blocks leaving (i) a corner cube empty or (ii) the middle cube empty.

[Justify your answers clearly.] [20 marks]

4. (i) State the inclusion-exclusion formula. Hence obtain a formula for the number D_n of derangements of n objects.

Calculate D_n for $1 \leq n \leq 6$.

(ii) Count the number of permutations in $S(n)$ which are the product of three disjoint transpositions.

Count the number of permutations in $S(n)$ which are the product of two disjoint 3-cycles.

List the four different cycle types of the derangements of six objects. Find the number of permutations of each of these cycle types and check that they add up to D_6.

The first row of a Latin square is

$$1 \quad 4 \quad 5 \quad 6 \quad 2 \quad 3 \quad .$$

Say how many possibilities there are for the second row.

(iii) The first two rows of a Latin square are

$$\begin{array}{cccccc} 1 & 4 & 5 & 6 & 2 & 3 \\ 5 & 6 & 1 & 4 & 3 & 2 \end{array} \quad .$$

Use the rook polynomial of a suitable 6×6 board to find how many possibilities there are for the third row.

[20 marks]

5. Solve the following recurrence relations. In each case you should find an expression for a_n and for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 3a_{n+1} - 2a_n$, $a_0 = 1, a_1 = 4$.

(ii) $a_{n+2} = 4a_{n+1} - 4a_n + 1$, $a_0 = a_1 = 1$.

(iii) $a_{n+1} = 2a_n + 2^{n+1}$, $a_0 = 0$.

(iv) $a_{n+1} = 2a_n - 2n$, $a_0 = 1$.

(v) $a_{n+1} = 3a_n + (n+1)3^n$, $a_0 = 0$.

[20 marks]

6. Obtain a formula for the generating function $S(t) = \sum_{n=0}^{\infty} s_n t^n$, where s_n is the number of solutions of $n = a + 3b + 4c$ in non-negative integers.

Calculate $S(t)$ up to the term in t^9.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer n. Find also the function $P_m(n)$ which enumerates the number of partitions of n into parts of length $\leq m$.

Define the term *Ferrers graph.* Use Ferrers graphs to establish a bijection between partitions of n with at most m parts and partitions of n into parts of length $\leq m$.

Hence write down the generating function $R(t)$ which enumerates partitions of n with at most 4 parts.

Show that $R(t) = (1 - t^2)^{-1} S(t)$. Hence or otherwise find the number of partitions of 9 with at most 4 parts.

Show that the generating function for partitions of n with at most 3 parts is given by $(1 - t^4)R(t)$ and deduce that $t^4 R(t)$ is the generating function for partitions of n with exactly 4 parts.

Hence determine the number of partitions of n with exactly 4 parts for all $n \leq 9$, and exhibit the corresponding Ferrers graphs in the case $n = 8$.

[20 marks]

7. (a) Define the term *symmetric function.* For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

(b) (i) Obtain the equation with roots $\alpha + \beta,\ \beta + \gamma,\ \gamma + \alpha$ in terms of the elementary symmetric functions of α, β and γ.

(ii) Write d_r for the determinant

$$\det \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^r & \beta^r & \gamma^r \end{vmatrix}$$

Show that, for each $r \geq 3$, $\psi_r = d_r/d_1$ is a symmetric function of α, β and γ.

Show that $d_1 = (\alpha - \beta)(\beta - \gamma)(\alpha - \gamma)$. Hence find d_4 and express ψ_4 in terms of the elementary symmetric functions of α, β and γ.

[20 marks]