1. (a) Use the binomial expansion of $(x+y)^n$ to show that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots(-1)^n\binom{n}{n}=0.$$

Hence show that $\binom{n}{0}+\binom{n}{2}+\cdots=\binom{n}{1}+\binom{n}{3}+\cdots=2^{n-1}$ for $n\geq 1$.

(b) Give a combinatorial argument to show that $\binom{2n}{2}=2\binom{n}{2}+n^2$.

Suggest, without proof, a similar formula for $\binom{kn}{2}$.

(c) Write down the number of anagrams of the word WOOLAMOOLOO?

How many anagrams have no appearance of OO?

How many have just one appearance of OO?

(d) (i) Count the number of ways of selecting a subset of 3 numbers from $\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\}$ so that the product of the numbers is odd.

(ii) Count the number of ways of selecting a subset of 3 numbers from $\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\}$ so that the sum of the numbers is odd.

(iii) Let A be any selection of 6 numbers from $\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\}$. Show that there must always be two subsets of A each with at most 3 numbers, which have the same total. [20 marks]

2. (a) (i) Derive a formula for the number of ways of distributing r identical objects into n distinct containers.

Show why this formula counts the number of solutions to the equation

$$k_1+k_2+\cdots+k_n=r$$

for non-negative integers $k_1,\ldots,k_n$, and why it also counts the number of ways of selecting r items from a choice of n distinct varieties, where the order of selection is unimportant.

(ii) How many numbers between 1000 and 9999 have the sum of their digits equal to 9?

How many of these have no digit equal to 0?

(b) The code for a safe is a sequence of 12 symbols consisting of eight asterisks $(*)$, and four numbers from the set $\{1,2,3,4\}$.

How many possible codes are there:

(i) if all four numbers must be different?

(ii) with no restrictions on the choice of numbers?

(iii) if all four numbers must be different and there is at least one asterisk between each pair of numbers? [20 marks]

3. State Hall's selection theorem.

(a) A rectangular array of squares is given, from which a subset B of the squares is excluded.

(i) Show how the problem of constructing a perfect cover of the remaining squares by 2×1 tiles can be reformulated in terms of selecting distinct representatives for a suitable collection of sets, describing carefully what sets are to be used.

(ii) For each array below, where excluded squares are shown as ■, list a suitable collection of sets for this reformulation.

(iii) Using this collection of sets decide, with reasons, whether or not there is a perfect cover of the squares shown as □ by 2×1 tiles.

□ □ □ □ □ □	□ □ ■ ■ □ ■
□ □ □ □ ■ ■	□ □ □ ■ □ □
□ ■ ■ □ □ □	□ □ ■ ■ □ □
■ ■ □ □ ■ □	□ ■ □ □ □ □
■ □ □ □ □ □	□ □ □ ■ □ □

(b) A tour party arrives for a night at a busy seaside hotel, where they are allocated individual rooms by the management.

The initial arrangements turn out not to suit everybody. Some of the party want larger beds, others are upset by the lack of a sea view. Some complain that their rooms are too far from the restaurant, and yet others that they are too close to the bar, and want non-smoking rooms.

After some consultation the tour operator manages to collect details of the rooms that would be acceptable to each person, and draws up the following table.

Person	*Room*	*Person*	*Room*
Ann	$1, 2, 3, 6$	*Beth*	$4, 9$
Carole	$1, 4, 9$	*Diana*	$2, 3, 4, 5, 8$
Eleanor	$4, 7, 10$	*Frances*	$1, 7, 9$
Gerry	$1, 2, 4, 5, 6$	*Helen*	$4, 5, 6, 7$
Irene	$1, 4, 7$	*Jean*	$1, 9, 10$

.

(i) Find an allocation which will suit everyone except Jean. Show that it is not possible to allocate satisfactory rooms for the whole party.

(ii) To add to the operator's problems it appears that no provision had been made for the driver, who wants one of the smoking rooms $8, 10$.

Luckily Beth and her sister Carole are prepared to share one of the large rooms $4, 9$. Show how everyone's requirements can now be met.

[Justify your answers clearly.] [20 marks]

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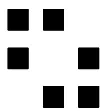
4. (i) Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated.

State the 'forbidden positions' formula.

State the inclusion-exclusion formula, and sketch briefly how this is used to prove the forbidden positions formula.

(ii) Calculate the rook polynomial of the 3×3 board shown.



(iii) The first two rows of a Latin square are

$$\begin{array}{cccccc} 1 & 4 & 5 & 6 & 2 & 3 \\ 2 & 1 & 6 & 3 & 4 & 5\ . \end{array}$$

Find how many possibilities there are for the third row. [20 marks]

5. Solve the following recurrence relations. In each case you should find an expression for a_n and for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 3a_{n+1} - 2a_n$, $a_0 = 1, a_1 = 4$.

(ii) $a_{n+2} = 4a_{n+1} - 4a_n + 1$, $a_0 = a_1 = 1$.

(iii) $a_{n+1} = 2a_n + 3^{n+1}$, $a_0 = 0$.

(iv) $a_{n+1} = 2a_n + 2^{n+1}$, $a_0 = 1$.

(v) $a_{n+1} = 3a_n + 2(n+1)$, $a_0 = 0$. [20 marks]

6. The Catalan numbers $\{c_n\}$ are defined recursively by the relations

$$c_0 = 1, c_{n+1} = \sum_{r=0}^{n} c_r c_{n-r}, \; n \geq 0.$$

(i) Write down the sequence of these numbers up to c_5.

(ii) Prove, by comparing the terms in x^{n+1}, that the generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

for the Catalan numbers satisfies the quadratic equation

$$C(x) = 1 + xC(x)^2.$$

(iii) Write a_n for the number of ways to join $2n$ points on the circumference of a circle in pairs by n straight lines drawn across the circle which do not intersect each other. Find a recurrence relation expressing a_{n+1} in terms of $a_i, i \leq n$ and deduce that $a_n = c_n$ for all $n \geq 1$.

(iv) Prove similarly that $c_n, \; n > 0$ counts the number of ways of cutting up a convex $(n+2)$-gon into n triangles by cuts from vertex to vertex.

(v) Using the equation $C(x) = 1 + xC(x)^2$ show that

$$xC(x) = \frac{1}{2}(1 - \sqrt{1-4x}),$$

and deduce that

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

[20 marks]

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7. Obtain a formula for the generating function $S(t) = \sum_{n=1}^{\infty} s_n t^n$, where s_n is the number of solutions of $n = a + 3b + 4c$ in non-negative integers.

Calculate $S(t)$ up to the term in t^{10}.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer n. Find also the function $P_m(t)$ which enumerates the number of partitions of n into parts of length $\leq m$.

Define the term *Ferrers graph*. Use Ferrers graphs to establish a bijection between partitions of n with at most m parts and partitions of n into parts of length $\leq m$.

Hence write down the generating function $R(t)$ which enumerates partitions of n with at most 4 parts.

Show that $R(t) = (1-t^2)^{-1}S(t)$ and hence find the number of partitions of 10 with at most 4 parts.

Show that $(1-t^4)R(t)$ enumerates partitions of n with at most 3 parts.

Use your calculations to determine the number of partitions of n with exactly 4 parts for all $n \leq 10$. Exhibit the corresponding Ferrers graphs in the case $n = 10$. [20 marks]

8. (a) Define the term *symmetric function*. For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

(b) Let α, β and γ be the roots of the equation $x^3 - 2x^2 - x + 5 = 0$.

(i) Write down their elementary symmetric functions, and find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$.

(ii) Find the cubic equation with roots $\alpha^2,\ \beta^2,\ \gamma^2$. [20 marks]