1. (a) Prove the following identities, either by algebraic manipulation or by combinatorial argument:

(i) $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$;

(ii) $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.

(b) How many ways are there to choose 13 cards from a standard deck such that at least one of the following is true: exactly 3 of them are diamonds, exactly 4 of them are hearts, exactly 5 of them are spades? (In this part and the next, you may express your answers in terms of binomial coefficients or factorials without expanding them.)

(c) In how many ways can the letters of the word "undefended" be rearranged? Of these, how many do not include the sequence NN? How many do not include the sequence EE?

[20 marks]

2. (a) Let a, b, c be positive integers such that $2b > a + c$. Let $m_1, \ldots, m_b$ be a sequence of positive integers whose sum is a. Prove that there is a subsequence consisting of consecutive terms $m_r, m_{r+1}, \ldots, m_s$ whose sum is c.

(b) Suppose given five points $P_1, \ldots, P_5$ in a square (including boundary) of side length 1. Prove that there are P_i, P_j with $i \neq j$ such that the distance from P_i to P_j is at most $1/\sqrt{2}$. Show how to choose $P_1, \ldots, P_5$ so that the minimum distance is exactly $1/\sqrt{2}$.

(c) Let $S = \{0, 1, \ldots, 22\}$ and let T be a subset of S with 7 elements. Prove that there are two distinct subsets of T with the same sum. [20 marks]

3. (a) State Hall's theorem. State a version of Hall's theorem that applies to matchings of two sets S and T in which each element of S must be paired with several elements of T.

(b) Eight people A, B, C, D, E, F, G, H are to do eight tasks $1, 2, \ldots, 8$. Only some of the people can do each task, as shown in the following table:

A	1, 3, 5	E	1, 3, 6
B	1, 4, 7, 8	F	3, 4, 8
C	2, 7, 8	G	5, 6
D	1, 5, 6	H	1, 5

(i) Show that seven of the tasks can be assigned, but not all eight.

(ii) Suppose it is decided that B will perform two of the tasks. Prove that it is now possible to assign all of the tasks. (One of the people will be idle.)

(iii) In the situation of (ii), is it possible to arrange for G not to be assigned a task?

[20 marks]

4. Four medical students $x_1, \ldots, x_4$ have applied for four hospital residencies $y_1, \ldots, y_4$. The students and the hospitals have the following preference orders:

$\mathbf{x_1}$	3	2	4	1	$\mathbf{y_1}$	1	3	4	2
$\mathbf{x_2}$	2	1	3	4	$\mathbf{y_2}$	4	1	3	2
$\mathbf{x_3}$	2	4	3	1	$\mathbf{y_3}$	2	4	3	1
$\mathbf{x_4}$	3	1	2	4	$\mathbf{y_4}$	1	3	2	4

(i) State the definition of a stable matching. Find (a) the stable matching that is optimal from the students' point of view, (b) the stable matching that is optimal from the hospitals' point of view.

(ii) Determine whether there is a stable matching in which x_4 is paired with (a) y_1; (b) y_4. [20 marks]

5. (i) Define a *rook polynomial*. State a formula that relates the rook polynomial of a board to that of its complement.

(ii) Hence, or otherwise, determine the rook polynomial of the 4×4 board shown here:

□	□	□	□
■	□	□	■
■	□	□	□
□	■	□	□

(where □ represents a square that belongs to the board and ■ one that does not).

(iii) Five people A, B, C, D, E are to be assigned to five offices $1, 2, 3, 4, 5$, subject to the following constraints: A and B are unwilling to accept office 1, B refuses office 2, C and E do not want office 3, D rejects office 4, and everyone is happy to accept office 5. In how many ways can these people be assigned offices?

[20 marks]

6. (a) Solve each of the following recurrences or systems of recurrences. In each case give explicit formulas both for the sequence a_i and the generating function $\sum a_i t^i$.

(i) $a_0 = 2, a_1 = 2, a_i = 4a_{i-1} - 4a_{i-2}$ for $i > 1$;

(ii) $a_0 = 1, b_0 = -1, a_i = 2a_{i-1} + b_{i-1}$ for $i > 0$, $b_i = a_{i-1} + 2b_{i-1}$ for $i > 0$. (Solve only for the a_i, not for the b_i.)

(iii) $a_0 = 1, a_i = 3a_{i-1} + 2^i$ for $i > 0$.

(b) For all positive integers i, let M_i be an $n \times n$ matrix with 2 on the diagonal, 1 immediately above it, and -1 immediately below it. For example,

$$M_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $a_i = \det M_i$ and $a_0 = 1$. Give an explicit formula for a_i and for $\sum a_i t^i$.

[20 marks]

7. Give generating functions for (a) the number of partitions of n with no even part repeated, (b) the number of partitions of n with no part a multiple of 4, (c) the number of partitions of n with no part repeated more than three times, and show that they are all equal.

[20 marks]

8. (i) Let a, b, c be complex numbers such that

$$\begin{aligned} a+b+c &= 3 \\ a^2+b^2+c^2 &= 5 \\ a^3+b^3+c^3 &= 7. \end{aligned}$$

Determine the cubic polynomial whose roots are a, b, c. Find $a^4+b^4+c^4$ and $a^5+b^5+c^5$ (preferably by using Newton's identities).

(ii) With a, b, c as above, determine the cubic polynomial whose roots are $a+b, b+c, c+a$.

[20 marks]