**1.** (a) A sports league contains 16 teams. Each team plays each other team, at home and away. How many games does each team play? In how many ways can the 16 teams be formed into 4 groups of 4 teams each?

The first 3 teams at the end of the season are awarded a gold, silver and bronze cup respectively. What is the number of ways the cups can be allocated? Three teams are relegated each season: what is the number of ways in which this can occur?

Nine of the teams are Northern, and seven are Southern. How are the answers to the preceding two questions affected if you know that 2 of the Northern teams win cups and just 1 Northern team is relegated?

(b) Let $S$ be a circle of radius $\sqrt{3}$ cm. Show that if 25 points are placed inside $S$, there must be two of them whose distance apart is at most 1 cm. (Hint: inscribe the circle in a hexagon.) [20 marks]

**2.** (a) Obtain a formula for the number of ways of distributing $r$ identical objects into $n$ distinct containers.

(b) How many terms are there in the expansion of $(v + w + x + y)^7$? Give the coefficient of $v^2x^3y^2$ in this expansion.

(c) Determine the number of integer solutions of $p + q + r + s = 45$ when:

(i) each of $p, q, r, s$ is non-negative;

(ii) each of $p, q$ is $\geq 6$ and $r \geq 7$, $s \geq 8$;

(iii) each of $p, q, r, s$ is positive and less than 16;

(iv) $5 \leq p \leq 19$, $6 \leq q \leq 19$, $4 \leq r \leq 19$ and $3 \leq s \leq 17$. [20 marks]

**3.** State Hall's Assignment Theorem.

(a) For each of the pruned chessboards shown below, construct a family of sets which has a system of distinct representatives if and only if there is a perfect cover by dominoes, and decide whether the chessboard has a perfect cover or not.

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Here a ■ denotes a square that has been deleted.

(b) Ann's mummy is giving a Xmas party and invites Ann's friends Ben, Cathy, David, Elli, Fred, Gina, Henry, Irene and John. She buys 10 presents and wraps them up, with a number on each, and at the end of the party gives presents 1-10 to the children in the order listed. When the presents are unwrapped, 3 of the children burst into tears because they want different presents. On enquiry, it is found that Ann would be happy with any of presents 1,3,6; Ben with 2,6,9; Cathy with 3,4,9; David with 2,4,5, Elli with 3,5,7, Fred with 1,4,8, Gina with 3,7,8, Henry with 2,3,4, Irene with 5,9,10 and John with 3,4,5. How can all the children be satisfied with the fewest possible number of presents being handed across to a different child?
[20 marks]

**4.** Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

If the first two rows of a Latin square are 123456 and 264531, how many possibilities are there for the third row?

If you also know that the third row is 615342, how many possibilities are there for the fourth row? [20 marks]

**5.** Solve the following recurrence relations. In each case you should find an expression for $a_n$ and also an expression for the generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

(i) $a_{k+2} = 3a_{k+1} - 2a_k$, $a_0 = 1, a_1 = 4$.

(ii) $a_{k+2} = 3a_{k+1} - 2a_k + 1$, $a_0 = 1, a_1 = 4$.

(iii) $a_{k+2} = 3a_{k+1} - 2a_k + 2^k$, $a_0 = 0, a_1 = \frac{1}{2}$.

(iv) $a_{k+1} = 3a_k + 3^k$, $a_0 = 0$.

(v) $a_{k+1} = 3a_k + (k+1)3^k$, $a_0 = 0$.

[20 marks]

**6.** (a) The function $\phi_k$ is the polynomial of degree $k+1$ such that for $n$ a positive integer, $\phi_k(n) = \sum_{r=1}^{n} r^k$. Obtain a formula for

$$F_n(x) := \frac{1}{2} + \sum_{k=0}^{\infty} \phi_k(n) \frac{x^k}{k!}.$$

Hence show that $-F_n(-x) = F_{-(n+1)}(x)$. Deduce that, for all positive integers $k$ and all $n$, $\phi_k(-n-1) = (-1)^{k-1}\phi_k(n)$, and hence that if $k$ is even, $\phi_k(n)$ is divisible by $(n + \frac{1}{2})$.

(b) State the inclusion-exclusion principle. Determine how many permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are such that no two consecutive numbers are divisible by 3. [20 marks]

**7.** (a) Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer $n$. Find also the function $P_o(n)$ which enumerates the number of partitions of $n$ into odd parts, and the function $P_d(t)$ to enumerate the partitions of $n$ with distinct parts. For any positive integer $n$, show that the number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.

Define Ferrers graphs.

(b) By enumerating partitions with distinct parts according to the number of parts, and comparing with the formula found in (a), obtain the identity

$$\prod_{1}^{\infty}(1+t^r) = \sum_{m=0}^{\infty} \frac{t^{\frac{1}{2}m(m+1)}}{(1-t)(1-t^2)\ldots(1-t^m)}.$$

[20 marks]

**8.** Define the term *symmetric function.* For any positive integer $n$, define the *elementary symmetric function* $\sigma_n$ and the *power sum symmetric function* $\pi_n$. State and prove the Newton Identities.

(a) By treating the first 4 identities as linear equations in $1, \pi_1, \pi_2, \pi_3$ and $\pi_4$, solve to express $\pi_4$ as a determinant in terms of the elementary symmetric functions, and evaluate the determinant.

(b) Given that the roots of the equation $t^3 - at^2 + bt - c = 0$ are $x_1, x_2$ and $x_3$, find the equation whose roots are $x_1x_2$, $x_1x_3$ and $x_2x_3$ and hence express the elementary symmetric functions of these three in terms of those of $x_1, x_2$ and $x_3$. [20 marks]